

Nr. 94
25. Apr. 2023

Preprint-Series: Department of Mathematics - Applied Mathematics

A complementary l1-TV reconstruction algorithm for
limited data CT

S. Göppel, J. Friel, M. Haltmeier

AppliedMathematics

Technikerstraße 13 - 6020 Innsbruck - Austria
Tel.: +43 512 507 53803 Fax: +43 512 507 53898
<https://applied-math.uibk.ac.at>

A complementary ℓ^1 -TV reconstruction algorithm for limited data CT

Simon Göppel

Department of Mathematics, University of Innsbruck
Technikerstrasse 13, 6020 Innsbruck, Austria
E-mail: simon.goeppel@uibk.ac.at

Jürgen Friel

Department of Computer Science and Mathematics, OTH Regensburg
Galgenbergstraße 32, 93053 Regensburg, Germany
E-mail: juergen.friel@oth-regensburg.de

Markus Haltmeier

Department of Mathematics, University of Innsbruck
Technikerstrasse 13, 6020 Innsbruck, Austria
E-mail: markus.haltmeier@uibk.ac.at

April 25, 2023

Abstract

In a variety of tomographic applications, data cannot be fully acquired, leading to severely underdetermined image reconstruction. Conventional methods result in reconstructions with significant artifacts. In order to remove these artifacts, regularization methods have to be applied that incorporate additional information. An important example is TV reconstruction which is well known to efficiently compensate for missing data and well reduces reconstruction artifacts. At the same time, however, tomographic data is also contaminated by noise, which poses an additional challenge. The use of a single regularizer within a variational regularization framework must therefore account for both the missing data and the noise. However, a single regularizer may not be ideal for both tasks. For example, the TV regularizer is a poor choice for noise reduction over different scales, in which case ℓ^1 curvelet regularization methods work well. To address this issue, in this paper we introduce a novel variational regularization framework that combines the advantages of two different regularizers. The basic idea of our framework is to perform reconstruction in two

stages, where the first stage mainly aims at accurate reconstruction in the presence of noise, and the second stage aims at artifact reduction. Both reconstruction stages are connected by a data proximity condition. The proposed method is implemented and tested for limited-view CT using a combined curvelet-TV approach. We define and implement a curvelet transform adapted to the limited view problem and demonstrate the advantages of our approach in a series of numerical experiments in this context.

Keywords: Image reconstruction, limited data, artifact reduction, sparse regularization, complementary image reconstruction, wedge-adapted curvelets.

1 Introduction

Limited data computed tomography (CT) is a prerequisite for a wide range of applications such as digital breast tomosynthesis, dental tomography and non-destructive testing. In this case, the available data is only a subset of the full data that would be required to uniquely identify the scanned object. Due to the lack of available scans, certain image features are invisible and important information may be obscured by artifacts generated during reconstruction [25, 26]. Although the characterization of limited view artifacts has been well researched [2, 14, 15], effective artifact reduction or compensation for missing data is still a challenge. This is even more true when the tomographic data is noisy, which creates additional hurdles.

Mathematically, limited-data CT can be written as an inverse problem of the form

$$v^\delta = \mathbf{K}_I u + \eta^\delta, \quad (1.1)$$

where $u \in L^2(\mathbb{R}^2)$ is the unknown image to be recovered, \mathbf{K}_I denotes the Radon transform with restricted angular range I and η^δ denotes the noise in the data. While the inverse problem of recovering an image from noisy CT measurements with complete data is already ill-posed [23], the reconstruction problem in the presence of missing data is severely under-determined. Direct methods such as filtered back projection (FBP) are sensitive to noise and do not handle missing data well, leading to typical limited data artifacts. To account for noise and missing data, further information that is available about the object to be recovered must be incorporated. Specific methods are therefore required that can both reliably remove noise and avoid artifacts caused by limited data.

One of the most successful approaches to problems of the form (1.1) is variational regularization, in which a stable and robust solution $u_\alpha^\delta \in L^2(\mathbb{R}^2)$ of the limited-data image reconstruction problem is determined as a minimizer of

$$\mathcal{T}_\alpha(u, v^\delta) = \frac{1}{2} \|\mathbf{K}_I u - v^\delta\|^2 + \alpha \mathcal{R}(u), \quad (1.2)$$

where \mathcal{R} is a suitable regularizer incorporating prior information and $\|\mathbf{K}_I u - v^\delta\|^2/2$ is the least squares data fitting functional. The variational approach offers great flexibility.

In particular, it can be adapted to the forward problem, the signal class, and the noise. For example, total variation $\mathcal{R}(u) = |u|_{\text{TV}}$ has been shown to be a good prior to complete missing data [24, 31, 35, 36]. On the other hand, the ℓ^1 -norm $\mathcal{R}(u) = \|\Psi u\|_1$ of wavelet or curvelet coefficients $\Psi u \in \Theta$, where usually one takes $\Theta = \ell_2(\Lambda, \mathbb{R})$ with suitable index set Λ , has been shown to be statistically optimal for Radon inversion from complete data [5]. On the downside, the mono-scale nature of the total variation does not lead to an optimal reconstruction in the presence of noise [18] and ℓ^1 -reconstructions hardly extrapolate missing data [27, 28].

The individual advantages and disadvantages of specific regularizers have led to so-called hybrid methods that combine two different regularizers within the variational regularization framework (1.2). For example, hybrid ℓ^1 -TV methods [19, 33] use the regularizer $\mathcal{R}(u) = \alpha|u|_{\text{TV}} + \beta\|\Psi u\|_1$. Given the above strengths and limitations of each individual regularizer, this is particularly attractive for CT with noisy limited data. However, the single hybrid regularizer must again account for both, the limited data and the denoising problem, which is a challenging task. Unfortunately, a fixed hybrid regularizer cannot fully avoid the drawbacks of the individual terms. For example, the TV term still leads to over or under smoothing of certain scales in the visible range, while the curvelet part still tries to suppress intensity values of invisible coefficients. To avoid these negative impacts, it is necessary to adapt each regularizer to its actual purpose.

In this paper, we present a novel complementary ℓ^1 -TV algorithm that addresses both the limited data problem and the noise reduction problem. It is based on a modified regularization approach that selects a regularizer for each of the two tasks and combines them in a synergetic way through data-proximity. More precisely, the proposed iterative reconstruction method generates two reconstructions $\theta \in \Theta$ and $u \in L^2(\mathbb{R}^2)$ by alternately solving

$$\begin{aligned} \min_{\theta} & \|\mathbf{K}_I(\Psi^*\theta) - v^\delta\|^2/2 + \alpha\|\theta\|_1 + \mu\|\mathbf{K}_I(u - \Psi^*\theta)\|^2/2 \\ \min_u & \mathcal{R}(u) + \mu\|\mathbf{K}_I(u - \Psi^*\theta)\|^2/2. \end{aligned}$$

Here, the auxiliary reconstruction $\Psi^*\theta$ targets a noise suppressing reconstruction addressed by the sparsity term $\|\theta\|_1$. The main reconstruction u performs data completion based by updating $\Psi^*(\theta)$ using the regularizer $\mathcal{R}(u)$. A key element is the coupling of both reconstructions that requires $\|\mathbf{K}_I(u - \Psi^*\theta)\|^2$ to be small, which we refer to as data-proximity coupling. Although our concept is applicable to any image reconstruction problem with limited data, we focus on limited data CT for clarity. In addition, we propose several variations of the data-proximity coupling, which will be discussed later in the manuscript.

Note that our method is quite different from post-processing an original reconstruction. In the latter case, the data-proximity term $\|\mathbf{K}_I(u - \Psi^*\theta)\|^2$ is replaced by a proximity in image space $\|u - \Psi^*\theta\|^2$, which forces u to be close to $\Psi^*(\theta)$, making artifacts difficult to remove.

2 Background

Throughout this article, we will use the following notation. The Fourier transform of a function $u \in L^2(\mathbb{R}^2)$ is denoted by $\mathbf{F}u$, where $\mathbf{F}u(\xi) \triangleq \int_{\mathbb{R}^2} u(x) e^{-i\langle \xi, x \rangle} dx$ for integrable functions and extended to $L^2(\mathbb{R}^2)$ by continuity. We write $u^*(x) \triangleq \overline{u(-x)}$, where \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$. Recall that the Fourier transform converts convolution into multiplication. In particular, for $u, w \in L^2(\mathbb{R}^2)$, with $\mathbf{F}u \in L^\infty(\mathbb{R}^2)$, the convolution $u * w \in L^2(\mathbb{R}^2)$ is well-defined and given by $u * w = \mathbf{F}^{-1}((\mathbf{F}u) \cdot (\mathbf{F}w))$. Furthermore, $\mathbf{F}_2 u$ denotes the Fourier transform of $u \in L^2(\mathbb{S}^1 \times \mathbb{R})$ with respect to the second argument.

2.1 The Radon transform

The full data Radon transform maps any function $u \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ to the collection of its line integrals

$$\mathbf{K}u(\omega, s) \triangleq \int_{\omega^\perp} u(x + s\omega) dx \quad \text{for } (\omega, s) \in \mathbb{S}^1 \times \mathbb{R}.$$

Here $\mathbb{S}^1 = \{\omega \in \mathbb{R}^2 \mid \|\omega\| = 1\}$ is the unit circle and any line of integration $\{x \in \mathbb{R}^2 \mid \langle \omega, x \rangle = s\}$ is described by a unit normal vector $\omega \in \mathbb{S}^1$ and oriented distance s from the origin. As the pairs (ω, s) and $(-\omega, -s)$ describe the same line, a full angular range of the Radon transform is given by an arc covering 180° . The Radon transform can be extended to an unbounded densely defined closed operator $\mathbf{K}: \mathcal{D}(\mathbf{K}) \subseteq L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{S}^1 \times \mathbb{R})$ with domain of definition $\mathcal{D}(\mathbf{K}) \triangleq \{u \in L^2(\mathbb{R}^2) \mid \|\cdot\|^{-1/2} \mathbf{F}u \in L^2(\mathbb{R}^2)\}$; see [32].

Lemma 2.1 (Fourier slice theorem). *For all $u \in \mathcal{D}(\mathbf{K})$ we have $\mathbf{F}_2(\mathbf{K}u)(\omega, \sigma) = \mathbf{F}u(\sigma\omega)$.*

In Lemma 2.1 and below $\mathbf{F}_2 v(\omega, \sigma) \triangleq \int_{\mathbb{R}} v(\omega, s) e^{-i\sigma s} ds$ the 1D Fourier transform of $v \in L^2(\mathbb{S}^1 \times \mathbb{R})$ in the second component. The Fourier slice theorem states that for any $\omega \in \mathbb{S}^1$, the Fourier transform of the Radon transform of some function in the second component equals the Fourier transform of that function along the Fourier slice $\{\sigma\omega \mid \sigma \in \mathbb{R}\}$.

Opposed to the full data case, in limited data CT, the Radon transform is only known on a certain subset. Equivalently, we may model limited view data with a binary mask as we will do here. For any subset $A \subseteq \mathbb{S}^1 \times \mathbb{R}$ we denote by χ_A the indicator function defined by $\chi_A(\omega, s) = 1$ if $(\omega, s) \in A$ and $\chi_A(\omega, s) = 0$ otherwise.

Definition 2.2. *For any $I \subseteq \mathbb{S}^1$ we define the limited data Radon transform as*

$$\begin{aligned} \mathbf{K}_I: \mathcal{D}(\mathbf{K}) \subseteq L^2(\mathbb{R}^2) &\rightarrow L^2(\mathbb{S}^1 \times \mathbb{R}) \\ u &\mapsto \chi_{I \times \mathbb{R}} \cdot (\mathbf{K}u). \end{aligned}$$

By the Fourier slice theorem, limited CT data is in one-to-one correspondence with the Fourier transform $\mathbf{F}u$ restricted to the set $W_I \triangleq \{\sigma\omega \mid \sigma \in \mathbb{R} \wedge \omega \in I\}$. We will call W_I the

visible wavenumber set, as only Fourier coefficients for wave numbers in W_I are provided by the data. Accordingly, we call $\mathbb{R}^2 \setminus W_I$ the invisible wavenumber set. We see that if $\mathbb{R}^2 \setminus W_I$ has non-vanishing measure, then \mathbf{K}_I has non-vanishing kernel consisting of all functions $u \in \mathcal{D}(\mathbf{K}_I)$ with $\text{supp } \mathbf{F}u \subseteq W_I$.

In limited view CT the set W_I forms a wedge, whereas in the sparse view case the set W_I forms a fan; see the left two images in Figure 2.1.

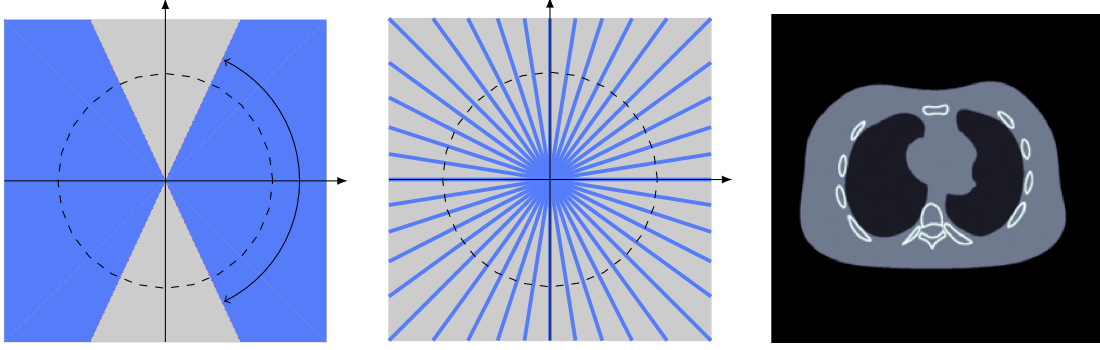


Figure 2.1: Left: Visible wavenumbers (blue) for limited view data covering 130° . Middle: Visible wavenumbers (blue) for sparse angular sampling using 20 angles. Right: Original NCAT phantom used for the numerical simulations shown below.

2.2 Frames and TI frames

In this article, we often use that the desired signal u allows a sparse representation in a suitable frame. In particular, we work with curvelet frames, which give an optimal sparse representation of cartoon-like images [5]. The same is true for shearlets [20]. Curvelets and shearlets form frames of $L^2(\mathbb{R}^2)$, and this section provides the necessary background.

2.2.1 Translational-invariant (TI) frames

Let Λ be an at most countable index set. A family $(\psi_\lambda)_{\lambda \in \Lambda}$ in $L^2(\mathbb{R}^2)$ is called a translation invariant frame (TI-frame) for $L^2(\mathbb{R}^2)$ if $\mathbf{F}\psi_\lambda \in L^\infty(\mathbb{R}^2)$ for all $\lambda \in \Lambda$ and

$$\forall u \in L^2(\mathbb{R}^2): \quad A\|u\|^2 \leq \sum_{\lambda \in \Lambda} \|\psi_\lambda * u\|^2 \leq B\|u\|^2, \quad (2.1)$$

for some $A, B > 0$. A TI-frame is called tight if $A = B = 1$. Because $\mathbf{F}\psi_\lambda \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\mathbf{F}u \in L^2(\mathbb{R}^2)$, we have $\psi_\lambda * u = \mathbf{F}^{-1}((\mathbf{F}\psi_\lambda) \cdot (\mathbf{F}u))$. From Plancherel's theorem we get $\|\psi_\lambda * u\|^2 = 2\pi \int_{\mathbb{R}^2} |\mathbf{F}\psi_\lambda|^2 |\mathbf{F}u|^2$. The right inequality in the TI-frame property (2.1) therefore in particular implies $(\psi_\lambda * u)_{\lambda \in \Lambda} \in \ell^2(\Lambda, L^2(\mathbb{R}^2))$. For some background on TI frames see [22] and [16] for TI frames in the context of inverse problems.

Along with TI-frames, we will make use of the TI-analysis and TI-synthesis operators

respectively, which are defined by

$$\begin{aligned}\Psi &: L^2(\mathbb{R}^2) \rightarrow \ell^2(\Lambda, L^2(\mathbb{R}^2)): u \mapsto (\psi_\lambda * u)_{\lambda \in \Lambda} \\ \Psi^* &: \ell^2(\Lambda, L^2(\mathbb{R}^2)) \rightarrow L^2(\mathbb{R}^2): (\theta_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} \psi_\lambda^* * \theta_\lambda.\end{aligned}$$

Note that the TI-analysis operator and the TI-synthesis operator are the adjoint of each other. The composition $\Psi^*\Psi$ is known as the TI-frame operator. Using the definition of the TI-analysis operator we can rewrite the frame condition (2.1) as $A\|u\|^2 \leq \|\Psi u\|^2 \leq B\|u\|^2$ for $u \in L^2(\mathbb{R}^2)$. The right inequality in (2.1) states that the TI-analysis operator Ψ is a well-defined bounded linear operator. The left inequality states that Ψ is bounded from below, that is, the pseudo-inverse $\Psi^\dagger \triangleq (\Psi^*\Psi)^{-1}\Psi^*$ is continuous.

2.2.2 Regular frames

Regular frames use inner products instead of convolutions as in TI frames for defining coefficients. Let Λ be an at most countable index set. A family $(\psi_\lambda)_{\lambda \in \Lambda}$ in $L^2(\mathbb{R}^2)$ is called a frame for $L^2(\mathbb{R}^2)$ if

$$\forall u \in L^2(\mathbb{R}^2): \quad A\|u\|^2 \leq \sum_{\lambda \in \Lambda} |\langle \psi_\lambda, u \rangle|^2 \leq B\|u\|^2, \quad (2.2)$$

for some $A, B > 0$. A frame is called tight if $A = B = 1$. In some sense the TI frame can be seen as a frame with index $\Lambda \times \mathbb{R}^2$. Note however that clearly the TI frame not a regular frame because the set $\Lambda \times \mathbb{R}^2$ is uncountable. Similar to the TI case, the analysis and synthesis operators of a regular frame are defined by

$$\begin{aligned}\Psi &: L^2(\mathbb{R}^2) \rightarrow \ell^2(\Lambda): u \mapsto (\langle \psi_\lambda, u \rangle)_{\lambda \in \Lambda} \\ \Psi^* &: \ell^2(\Lambda) \rightarrow L^2(\mathbb{R}^2): (\theta_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} \psi_\lambda^* \theta_\lambda\end{aligned}$$

and the composition $\Psi^*\Psi$ is the frame operator.

Under suitable regularity assumptions [10, 22], a regular frame with index set $\lambda \times \mathbb{Z}^2$ can be obtained from a TI frame with index set λ by discretizing the convolution in (2.1). For multiscale systems such as wavelets of curvelets, the associated λ -dependent subsampling destroys translation invariance, which can lead to degraded performance and reconstruction. The advantages of the TI system for simple denoising tasks have been investigated in [8], and its regularization properties for inverse problems in [16].

2.3 Variational image reconstruction

A practically successful and theoretically well analyzed method for solving inverse problems of the form (1.1) is variational regularization. Here, the available prior information is

incorporated by a regularization functional $\mathcal{R}: L^2(\mathbb{R}^2) \rightarrow [0, \infty]$ and an approximate image is recovered by minimizing the Tikhonov functional $\mathcal{T}_\alpha(u, v^\delta) = \|\mathbf{K}_I u - v^\delta\|^2/2 + \alpha\mathcal{R}(u)$ with respect to u ; see (1.2).

Variational regularization is well-posed, stable and convergent in the following sense: (i) $\mathcal{T}_\alpha(\cdot, v^\delta)$ has a minimizer u_α^δ ; (ii) minimizers depend continuously on data v^δ ; (iii) if $\|v - v^\delta\| \leq \delta$ with $v \in \text{ran}(\mathbf{K}_I)$ and $\alpha = \alpha(\delta)$ is selected properly then u_α^δ converges (as $\delta \rightarrow 0$) to an \mathcal{R} -minimizing solution of $\mathbf{K}_I u = v$ defined by

$$\min_u \mathcal{R}(u) \quad \text{such that } \mathbf{K}_I u = v. \quad (2.3)$$

These properties hold true under the assumption that \mathcal{R} is convex, weakly lower semicontinuous and coercive [29]. The characterization (2.3) of the limiting solutions reveals two separate tasks to be performed by the regularizer: Besides noise-robust reconstructions via minimization of the Tikhonov functional, it also serves as criteria for selecting a particular solution. Obviously, it is difficult to optimally perform both tasks with a single regularizer.

Note that the selection of a particular solution via (2.3) addresses the non-uniqueness and implicitly performs data completion. This is equivalent to the selection of the component of the reconstruction in the kernel $\ker(\mathbf{K}_I)$. While the standard Hilbert space norm regularizer completes the missing data with zero, other regularizers perform non-zero data completion. The data completion strongly depends on the chosen regularizer.

While there are many reasonable choices for the regularizer \mathcal{R} , in this paper we will mainly focus on the ℓ^1 -norm with respect to a suitably chosen frame and the total variation, each one coming with its own benefits and shortcomings.

2.3.1 Sparse ℓ^1 -regularization

Let Ψ^* denote the synthesis operator of a frame and set $\Psi^\dagger \triangleq (\Psi^* \Psi)^{-1} \Psi^*$. In particular, any $u \in L^2(\mathbb{R}^2)$ can be written as $u = \Psi^\dagger \Psi u$. Synthesis sparsity that $u = \Psi^* \theta$ where θ has only a few non-vanishing entries, whereas analysis sparsity refers to Ψu having only few non-vanishing entries. Sparsity can be implemented via regularization using the ℓ^1 norm. There are at least two different basic instances of sparse ℓ^1 -regularization namely the synthesis and analysis formulations

$$f_{\alpha, \delta}^{\text{ana}} = \arg \min_u \frac{1}{2} \|\mathbf{K}_I u - v^\delta\|^2 + \alpha \|\Psi u\|_1 \quad (2.4)$$

$$\theta_{\alpha, \delta}^{\text{syn}} = \arg \min_\theta \frac{1}{2} \|\mathbf{K}_I (\Psi^* \theta) - v^\delta\|^2 + \alpha \|\theta\|_1, \quad (2.5)$$

with $f_{\alpha, \delta}^{\text{syn}} \triangleq \Psi^* (\theta_{\alpha, \delta}^{\text{syn}})$. Synthesis and analysis regularization are equivalent in the basis case where they can be explicitly computed via the diagonal frame decomposition [11, 16]. In the general case synthesis regularization, analysis regularization and regularization via

the diagonal frame decomposition are however fundamentally different [13].

Frame based sparsity constraints have been widely employed for various reconstruction tasks [3, 4, 34]. Note that theoretical and practical issues for general variational regularization can in particular be applied to ℓ^1 -regularization. Additionally, ℓ^1 -regularization comes with additional recovery guarantees both in the deterministic and statistical context [5, 17].

2.3.2 TV regularization

Total variation regularization is a special case of variational regularization [1, 29] where the regularizer in (1.2) is taken as the total variation (TV)

$$|u|_{\text{TV}} \triangleq \sup \left\{ \int_{\mathbb{R}^2} u \operatorname{div} v \mid v \in \mathcal{C}_c^1(\mathbb{R}^2, \mathbb{R}^2) \wedge \|v\|_{2,\infty} \leq 1 \right\},$$

where $\|v\|_{2,\infty} \triangleq \sup_x (v_1(x)^2 + v_2(x)^2)^{1/2}$. TV regularization has been proven to well account for missing data in CT image reconstruction [24, 31].

Adding the TV semi-norm to the data-fidelity-objective function tends to smooth out noise in the image while preserving edges within the image. However as for other mono-scale approaches, there is a trade-off between noise reduction and preserving features at specific scales. Natural images have features across multiple scales which become either over or under smoothed depending on the particular choice of the regularization parameter [6, 18]. This already has negative impact for fully sampled tomographic systems or simple denoising. To account for the noise a sufficiently large regularization parameter is required that at the same time removes structures at small scales.

2.3.3 Hybrid regularizers

Hybrid methods aim to combine benefits of an ℓ^1 regularizer and an additional regularizer such as the TV-seminorm and minimize functionals of the type

$$\mathcal{T}_{\alpha,\beta}^{\text{hybrid}}(u, v^\delta) = \frac{1}{2} \|\mathbf{K}_I u - v^\delta\|^2 + \alpha \|\Psi u\|_1 + \beta \mathcal{R}(u). \quad (2.6)$$

In that context, the sparsity promoting nature of $\|\cdot\|_1$ and the data completion property of $\mathcal{R} = |\cdot|_{\text{TV}}$ are utilized. The ℓ^1 -term targets a noise-reduced reconstruction and the \mathcal{R} -term targets artifact reduction.

Various forms of hybrid ℓ^1 -TV regularization techniques have been proposed [19, 21, 34]. While these methods have been shown to outperform both pure TV and pure ℓ^1 regularization, they still carry the drawbacks of both approaches.

Minimizing (2.6) has the drawback that the ℓ^1 -penalty and the TV penalty work against each other in the following sense. The ℓ_1 -norm enforces sparsity of the reconstructed coefficients and for that purpose seeks to recover an image where missing data completed

by values close to zero. On the other hand, the strength of TV is to add missing data in a non-vanishing matter. This can be clearly seen for plain inpainting problem where data is given by the restriction $v_I = u|_I$. If for example u is a constant image then filling the missing data with this constant results in minimal total variation. This however works against the sparsity constraint in a localized frame which aims to fill missing data with small intensity values.

3 Complementary ℓ^1 -TV reconstruction

We now describe our proposed framework which basically alternates between a reconstruction step and an artifact reduction step inspired by backward backward (BB) splitting. For the following let $\Psi^* : \Theta \rightarrow L^2(\mathbb{R}^2)$ be the synthesis operator of a frame (where $\Theta = \ell^2(\Lambda)$) or a TI-frame (where $\Theta = \ell^2(\Lambda, L^2(\mathbb{R}^2))$).

3.1 BB splitting algorithm

Actual implementation of variational regularization(1.2) requires iterative minimization. Splitting methods are very successful in that context. In particular, BB splitting applied to the hybrid approach (2.6) will be the starting point of our approach. Consider the splitting $\mathcal{T}_{\alpha,\beta}(u, v^\delta) = \mathcal{F}_\alpha(u, v^\delta) + \mathcal{G}_\beta(u)$ with

$$\begin{aligned}\mathcal{F}_\alpha(u, v^\delta) &\triangleq \frac{1}{2} \triangleq \|\mathbf{K}_I u - v^\delta\|^2 + \alpha \|\Psi u\|_1 \\ \mathcal{G}_\beta(u) &\triangleq \beta \mathcal{R}(u).\end{aligned}$$

Because $\mathcal{F}_\alpha(\cdot, v^\delta)$ and \mathcal{G}_β are both non-smooth, methods that treat both functionals implicitly are an appealing choice. For that purpose one can use the BB splitting algorithm which with coupling constant $\mu > 0$ and starting value $u^0 \in L^2(\mathbb{R}^2)$ reads

$$w^{n+1} \triangleq \arg \min_h \mathcal{F}_\alpha(w, v^\delta) + \frac{\mu}{2} \|w - u^n\|^2 \quad (3.1)$$

$$u^{n+1} \triangleq \arg \min_u \mathcal{G}_\beta(u) + \frac{\mu}{2} \|w^n - u\|^2. \quad (3.2)$$

The BB splitting algorithm is known to converge [9] to the minimizer of $\mathcal{F}_\alpha(\cdot, v^\delta) + \beta \mathcal{R}_\mu$ where \mathcal{R} is the Moreau envelope $\mathcal{R}_\mu(u) \triangleq \inf_w \mathcal{R}(u) + \mu \|u - w\|^2/2$ of the hybride regularizer.

The iterates of the BB splitting algorithm are noise-reduced near solutions of (1.1) because of the functional $\mathcal{F}_\alpha(w, v^\delta)$ in (3.1) and regular because of \mathcal{G}_β (3.2). The iterates w_n, u_n are coupled via the proximity measure $\|u - w\|^2/2$ resulting in two sequences that are close to each other in the reconstruction domain.

3.2 Proposed reconstruction framework

Our algorithm is motivated by the BB splitting iteration (3.1), (3.2) utilizing a synthesis version for ℓ^1 minimization and TV regularization for the regularizer \mathcal{R} . The main difference, however, to the BB iteration is that the proximity term $\|u - w\|^2/2$ in the iterative updates are replaced by the data-proximity coupling term $\|\mathbf{K}_I(u - w)\|^2/2$.

Our goal is to construct two sequences $(\theta_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$ such that $\Psi^*\theta_n$ as well u_n are approximate solutions of $\mathbf{K}_I u = v^\delta$, however targeting different particular solutions. The reconstruction $\Psi^*\theta_n$ is a noise reduced reconstructions and u_n is an updated version of $\Psi^*\theta_n$ targeting reduced limited data artifacts based on \mathcal{R} . To that end define the functionals

$$\begin{aligned} \mathcal{F}_\alpha(\theta, v^\delta) &\triangleq \frac{1}{2} \triangleq \|\mathbf{K}_I(\Psi^*\theta) - v^\delta\|^2 + \alpha\|\Psi u\|_1 \\ \mathcal{G}_\beta(u) &\triangleq u \mapsto \beta|u|_{\text{TV}} + \mathbf{1}_{\geq 0}, \end{aligned}$$

with $\mathbf{1}_{\geq 0}$ being the indicator function of the positive cone given by $\mathbf{1}_{\geq 0}(u) = 0$ if $u \geq 0$ and $\mathbf{1}_{\geq 0}(u) = \infty$ otherwise.

Image reconstruction is done in an iterative fashion similar to (3.1) however using the data-proximity coupling $\|\mathbf{K}_I(u - w)\|^2/2$. For that purpose we suggest the iterative procedure

$$\theta^{n+1} \triangleq \arg \min_{\theta} \mathcal{F}_\alpha(\theta, v^\delta) + \frac{\mu}{2} \|\mathbf{K}_I(u_n - \Psi^*\theta)\|^2 \quad (3.3)$$

$$u^{n+1} \triangleq \arg \min_u \mathcal{G}_{\beta(n)}(u) + \frac{\mu}{2} \|\mathbf{K}_I(u - \Psi^*\theta_n)\|^2, \quad (3.4)$$

with starting value $u^0 \in L^2(\mathbb{R}^2)$. Here $\|\mathbf{K}_I(u - \Psi^*\theta)\|^2/2$ is the data-proximity coupling term and $\mu, \alpha, \beta(n) > 0$ are parameters. The resulting complementary ℓ^1 -TV reconstruction procedure is summarized in Algorithm 1.

Algorithm 1 Proposed complementary ℓ^1 -TV minimization

Choose $\mu, \alpha, \beta(n) > 0$ and $N \in \mathbb{N}$

Initialize $f_0 \leftarrow 0$ and $n \leftarrow 0$

repeat

$\theta_{n+1} \leftarrow \arg \min_{\theta} \mathcal{F}_\alpha(\theta, v^\delta) + \mu \|\mathbf{K}_I(u_n - \Psi^*\theta)\|^2/2$

$u_{n+1} \leftarrow \arg \min_u \mathcal{G}_{\beta(n)}(u) + \mu \|\mathbf{K}_I(u - \Psi^*\theta_n)\|^2/2$

$n \leftarrow n + 1$

until $n \geq N$

The proposed steps (3.3), (3.4) in Algorithm 1 come with a clear interpretation. The first step (3.3) is a sparse ℓ^1 -reconstruction scheme with good noise handling capabilities. The second step minimizes the TV norm with the penalty $\|\mathbf{K}_I(u - \Psi^*\theta)\|^2/2$ and targets artifact reduction. Note that the number N of outer iterations in Algorithm 1 as well as the parameters $\mu, \alpha, \beta(n)$ have influence on the final performance. The theoretical analysis is interesting and challenging but beyond the scope of this paper.

4 Numerical Experiments

In this section we present numerical results using the proposed Algorithm 1 and compare it with standard filtered back projection (FBP), ℓ_1 -synthesis regularization (2.5), TV regularization and hybrid ℓ^1 -TV regularization (2.6). We consider a limited view as well as a sparse angle scenario and use the NCAT phantom [30] as image to be recovered (see Figure 2.1). The NCAT phantom resembles a thorax CT scan, with the spine at the bottom, and ribs on the sides. The forward and adjoint Radon transforms are computed using Matlabs standard functions. To mimic real life applications we perturbed the data by various Poisson noise corresponding to 10^a incident photons per pixel bin with $a = 3, 4, 5$.

4.1 Implementation details

All minimization problems are solved with the Chambolle-Pock algorithm [7] using 200 iterations for ℓ^1 minimization, and 500 iterations for TV and hybrid ℓ^1 -TV minimization. This was also the case for the complementary approach, where for 10^5 and 10^4 photon counts we chose $N = 10$ and for 10^3 photon counts we chose $N = 4$ outer iterations. We take the n -th initial value for the θ and u update as θ_{n-1} and u_{n-1} , respectively. For Ψ we use a self-designed TI curvelet transform that in the case of limited view data is adapted to the visible wedge; see Appendix A. Total variation is implemented as the $(2, 1)$ -norm of the discrete gradient computed with finite differences.

The regularization parameters for Algorithm 1 are optimized for μ, α, β with $\beta(n) = 2^n \beta$. Since the described reconstruction techniques rely on good choices for regularization parameters α, β, μ we perform systematic parameter sweeps in all cases to obtain optimal reconstructions and a fair comparison. The parameters were optimized in terms of the relative ℓ^2 reconstruction error $\|u_{\text{rec}} - u\|_2 / \|u\|_2$, where u is the true signal and u_{rec} the reconstruction. For each parameter and method, we performed a 1D grid search to obtain the lowest ℓ^2 reconstruction error. In particular, for the proposed complementary ℓ^1 -TV algorithm, we first determine the optimal parameter α , and used the optimal choice of the θ -update as input for the optimization of the parameter β . All subsequent iterations were then calculated using these parameters.

For limited view experiments, we chose angular sampling points $\omega(\phi) = (\cos(\phi), \sin(\phi))$ with $\phi = -65^\circ, \dots, 64^\circ$ resulting in a total number of 130 directions covering an angular domain of 130° . For the sparse view problem we generate Radon data with an angular range of $[-90^\circ, 90^\circ)$, and a total number of 50 angular projections. Photon noise using 10^4 photon counts per bin was added to the data.

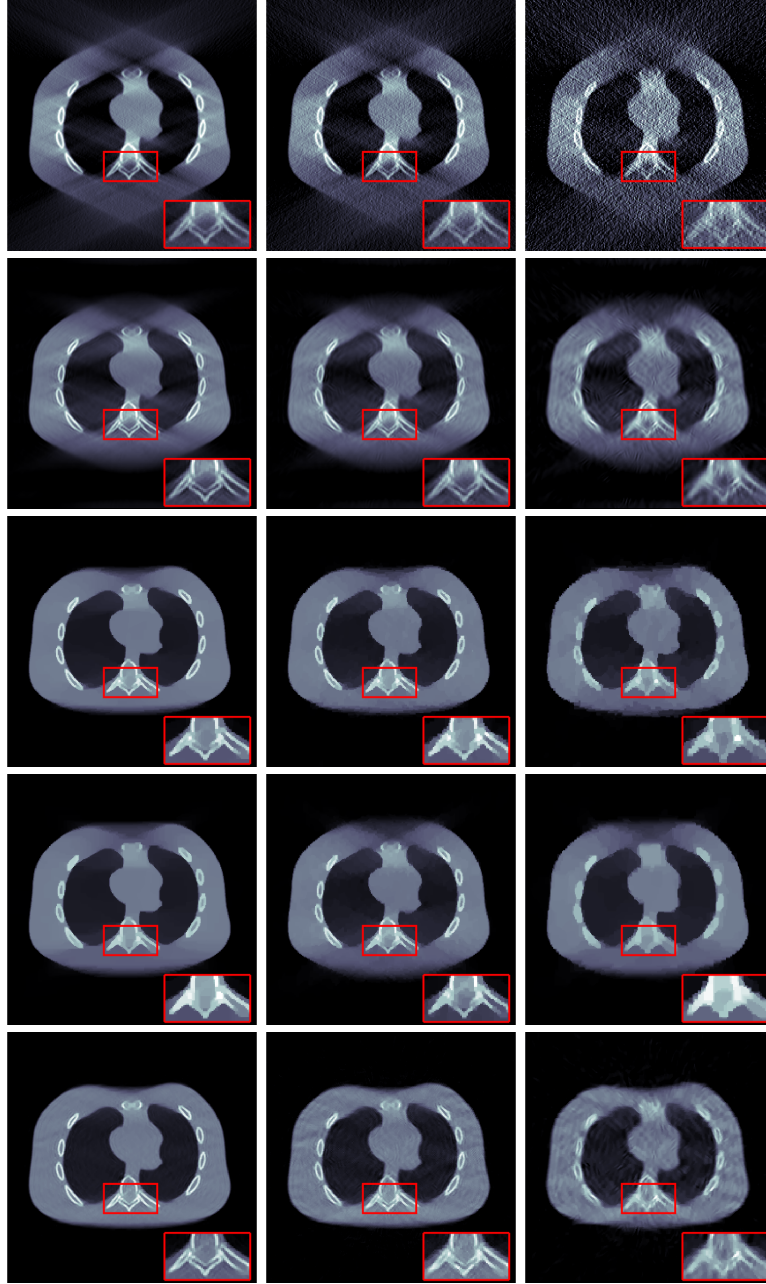


Figure 4.1: Reconstructions from limited view data. From top to bottom, each row shows results using FBP, ℓ^1 -curvelet reconstruction, TV reconstruction, hybrid ℓ^1 -TV and complementary ℓ^1 -TV regularization. Each column from left to right, corresponds to a different number of photons 10^5 , 10^4 and 10^3 . The pixel value range is set to $[0, 1]$ for all images.

4.2 Results for limited view data

Figure 4.1 shows reconstruction results for the limited view problem using FBP, ℓ^1 curvelet reconstruction, TV reconstruction, hybrid ℓ^1 -TV and the proposed complementary ℓ^1 -TV reconstruction. The results show that the complementary ℓ^1 -TV approach seems to combine the denoising and artifact removing properties of the regularizers in an optimal

way. Taking a closer look at the lowest noise level (10^5 photon counts) in the first column, we see that the FBP-reconstruction (top row) and the ℓ^1 -reconstruction (row 2) suffer from severe limited view artifacts. While the TV regularized (row 3) shows less artifacts, we find on the other hand that the fine details of the spine in the magnified part of the image are not reconstructed correctly anymore. This is typical for TV regularization when the regularization parameter has to be chosen too high in order to address the noise, resulting in block like artifacts. A similar observation holds true for the hybrid ℓ^1 -TV reconstruction (row 4). Taking a closer look at the results for the proposed algorithm (bottom row) we see that not only are we able to remove the limited view artifacts, but also to recover the fine details accurately. Furthermore, in comparison to the TV reconstruction we observe that the overall shape of the phantom is better approximated by our approach.

# photons	method	ℓ^2 -error	PSNR	SSIM
10^5	FBP	0.2496	17.1021	0.2693
	ℓ^1	0.0756	22.590	0.559
	TV	0.0187	29.725	0.953
	ℓ^1 -TV	0.0368	25.4124	0.8540
	proposed	0.0103	31.438	0.949
10^4	FBP	0.2719	16.7306	0.1635
	ℓ^1	0.0784	22.1291	0.5430
	TV	0.0246	27.1590	0.9210
	ℓ^1 -TV	0.0500	24.0859	0.7633
	proposed	0.0161	29.0141	0.8815
10^3	FBP	0.4961	14.1189	0.0696
	ℓ^1	0.0907	21.4974	0.4328
	TV	0.0411	24.9321	0.8621
	ℓ^1 -TV	0.0545	23.7100	0.7898
	proposed	0.0311	26.1420	0.7906

Table 1: Reconstruction errors for limited view reconstructions.

Similar conclusions can be drawn from the second column of Figure 4.1 showing results for 10^4 photon counts. Here for the TV and hybrid ℓ^1 -TV regularization even more details are lost. For the other methods, we still have a high level of details visible in the recovered images. However, only for the proposed method we also obtain an artifact free reconstruction. We attribute the remaining perturbations to the soft-thresholding procedure, that are part of the θ -update step. The last column in Figure 4.1 depicts the reconstructions for 10^3 photon counts (the highest noise level in our experiments). As we see, no method is able to recover the fine structures reliably anymore. However, note that for TV and hybrid ℓ^1 -TV regularization some of the ribs, which are boundaries of ellipse like structures, now appear to be filled. Simple curvelet- ℓ_1 regularization and the complementary ℓ^1 -TV approach still recover the fine holes inside these structures. Again, our method is capable of removing the limited view artifacts, while also being able to produce a good approximation to the

overall shape and details of the phantom.

Summarizing, we can say that our proposed algorithm combines the advantage of both, the denoising capabilities of curvelet- ℓ_1 regularization, the artifact removal and data recovery properties of the TV regularization approach. A quantitative comparison is given in Table 1 which compares the reconstructions in terms of the relative ℓ_2 -error, the peak signal-to-noise ratio (PSNR), as well as the structural similarity index measure (SSIM). The best values in each group are highlighted by bold letters. As we can see, the complementary ℓ^1 -TV approach produces the best reconstructions in terms of the ℓ^2 -error and PSNR, while simple TV regularization is optimal in terms of the SSIM. We find that quantitatively, TV regularization and the complementary ℓ^1 -TV approach are rather similar. However, qualitatively the advantages of the complementary ℓ^1 -TV method are clearly visible.

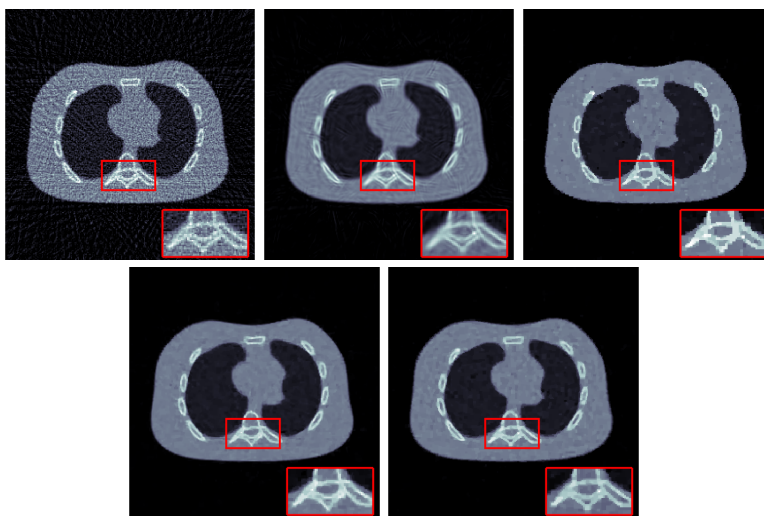


Figure 4.2: Reconstructions from sparse view data using FBP (top left), ℓ^1 -curvelet (top center), TV reconstructions (top right), hybrid ℓ^1 -TV (bottom left) and complementary ℓ^1 -TV (bottom right). The pixel value range is set to $[0, 1]$ for all images.

4.3 Results for sparse view data

Figure 4.2 shows reconstruction results for the sparse view problem using FBP, ℓ^1 curvelet reconstruction, TV reconstruction, hybrid ℓ^1 -TV and the proposed complementary ℓ^1 -TV regularization. We see that all reconstruction methods are able to reproduce the overall phantom rather good. Taking a closer look at the magnified details, we see that the ℓ^1 -curvelet reconstruction is able to image the spine rather good. However, we also see that the phantom also suffers from perturbations caused by the soft-thresholding of curvelet coefficients. The TV regularized reconstruction on one hand does not show severe artifacts, but on the other hand is not able to well recover fine details. Furthermore, some of the inner holes of the ribs start to become filled by the TV regularization, similar to the limited view case. The hybrid ℓ^1 -TV and the proposed complementary ℓ^1 -TV reconstruction on the other hand are able to incorporate both advantages from curvelet- ℓ_1 as well as TV

regularization. The spine is represented rather well and the phantom does not suffer from curvelet artifacts in both reconstructions.

# photons	method	ℓ^2 -error	PSNR	SSIM
10^4	FBP	0.1048	20.8702	0.1767
	ℓ^1	0.0136	29.7290	0.7308
	TV	0.0117	30.3933	0.9294
	ℓ^1 -TV	0.0080	32.0445	0.8884
	proposed	0.0101	31.0289	0.9302

Table 2: Reconstruction errors for sparse view reconstructions.

A quantitative error assessment is given in Table 2. Quantitatively, the hybrid ℓ^1 -TV method appears to perform slightly better than the other methods. The visual difference however is quite small and both methods produce equally good reconstructions, where the fine details in the phantom are well represented.

5 Conclusion

Similar to many other image reconstruction problems, limited-data CT suffers from instability regarding noise and non-uniqueness, leading to artifacts in image reconstruction. Common regularization approaches use a single regularizer to address both issues, which is accurate for one of the two tasks but not well adapted to the other. To address this issue, in this paper we propose a complementary ℓ^1 -TV algorithm that advantageously combines the denoising properties of ℓ^1 -curvelet regularization and the data completion properties of TV. The main ingredient of our procedure is data-proximity coupling instead of the standard image-space coupling.

There are many potential future research directions extending our framework. We can integrate the data-proximity coupling into other splitting type method using proximal terms such as the ADMM algorithm. Further, data-proximity coupling can be combined with preconditioning or other coupling terms. For example, one might replace $\|\mathbf{K}_I(u - \Psi^*\theta)\|^2$ by $\|\mathbf{P}_{\ker(\mathbf{K}_I)}(u - \Psi^*\theta)\|$ or may use hard constraints forcing $\mathbf{K}_I\Psi^*\theta = \mathbf{K}_Iu$. Further, one can also consider general discrepancy functionals \mathcal{F}_0 in place of the least squares functional $\|\mathbf{K}_Iu - v^\delta\|^2/2$. From the analysis side, studying convergence of iterative procedures as well as regularization properties is an important line of future research. Furthermore, a comprehensive investigation of TI frames for iterative regularization methods would be an interesting research focus. This includes a thorough analysis of theoretical properties along with numerical experiments. In particular, in combination with the limited view CT problem, the study of wedge adapted curvelets, and similar extensions to other limited data problem, could be of high interest.

Acknowledgments

The contribution by S. G. is part of a project that has received funding from the European Union’s Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 847476. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

A Wedge-adapted TI curvelet frames

Standard curvelets are not well adapted to limited angle data as some curvelets elements might may have small visible components. Our aim is therefore to construct a curvelet transform that is adapted to the limited view data \mathbf{K}_I where $I = \{(\cos \phi, \sin \phi) \mid \phi \in [-\Phi, \Phi]\}$ for some $\Phi < \pi/2$. The basic idea is to construct a specific partition of the frequency plane that respects the visible wedge $W_I = \mathbb{R}I$; see left image in Figure 2.1. We work with TI variants as the lack of translation invariance usually results in visual artifacts [22]. For a recent work on TI frames in the context of regularization theory see [16].

A.1 Standard TI curvelet frame

Consider the basic radial and angular Mayer base windows $W: [1/2, 2] \rightarrow [0, 1]$ and $V: [-1, 1] \rightarrow [0, 1]$

$$W(r) \triangleq \begin{cases} \cos((\pi/2)\nu(5-6r)) & \text{if } 2/3 \leq r \leq 5/6 \\ 1 & \text{if } 5/6 \leq r \leq 4/3 \\ \cos((\pi/2)\nu(3r-4)) & \text{if } 4/3 \leq r \leq 5/3 \\ 0 & \text{otherwise,} \end{cases}$$

$$V(\phi) \triangleq \begin{cases} 1 & \text{if } |\phi| \leq 1/3 \\ \cos((\pi/2)\nu(3|\phi|-1)), & \text{if } 1/3 \leq |\phi| \leq 2/3 \\ 0 & \text{otherwise.} \end{cases}$$

Here, the auxiliary function ν is chosen to satisfy $\nu(0) = 0$, $\nu(1) = 1$ and $\nu(x) + \nu(1-x) = 1$. Possible choices are polynomials, for example $\nu(x) = 3x^2 - 2x^3$, $\nu(x) = 5x^3 - 5x^4 + x^5$ or $\nu(x) = x^4(35 - 84x + 70x^2 - 20x^3)$. Depending on the choice of ν , the angular windows have smaller or bigger overlap. In this paper we use $\nu(x) = \chi_{(0,1)}s(x-1)/(s(x-1) + s(x))$ with $s(x) = \exp(-(1+x)^{-2} - (1-x)^{-2})$.

The TI-curvelets are defined in the frequency space using products of rescaled versions of

the radial and angular base windows

$$\mathbf{F}\boldsymbol{\psi}_{j,\ell}(\xi) = 2^{3j/4}W(2^{-j}r)^2 \cdot V(2\pi\phi/N_j - \ell)^2, \quad (\text{A.1})$$

where $\xi = r(\cos \phi, \sin \phi)$ and $N_j \in \mathbb{N}$ and $\Lambda \triangleq \{(j, \ell) \mid j \in \mathbb{N} \wedge \ell \in \{-N_j/2, \dots, N_j/2 - 1\}\}$. At every at scale j the radial window $W(2^{-j}r)$ defines a ring that is further partitioned into N_j angular wedges $V(2\pi\phi/N_j - \ell)$.

Theorem A.1. $(\boldsymbol{\psi}_{j,\ell})_{(j,\ell) \in \Lambda}$ is a tight TI-frame.

Proof. From the definition of the basis windows we have $\sum_{\ell=-N_j/2}^{N_j/2-1} V(2\pi\phi/N_j - \ell)^2 = 1$ and $\sum_{j \in \mathbb{Z}} |W(2^{-j}r)|^2 = 1$ and therefore $\sum_{j,\ell} |\mathbf{F}\boldsymbol{\psi}_{j,\ell}(\xi)|^2 = 1$. By the Plancherel identity this is equivalent to the tight frame condition (2.1) with $A = B = 1$. \square

Curvelet frames are defined by sampling $\boldsymbol{\psi}_{j,\ell} * u$ at points $M_{j,\ell}k$ with a sampling matrix $M_{j,\ell} \in \mathbb{R}^{2 \times 2}$ and sampling index $k \in \mathbb{Z}^2$. Defining $\boldsymbol{\psi}_{j,\ell,k} := \boldsymbol{\psi}_{j,\ell}(x - M_{j,\ell}k)$, this results in curvelet coefficients $\boldsymbol{\psi}_{j,\ell} * u(M_{j,\ell}k) = \langle \boldsymbol{\psi}_{j,\ell,k}, f \rangle$. The family $(\boldsymbol{\psi}_{j,\ell,k})_{j,\ell,k}$ is a tight frame which the associated reproducing formula $u = \sum_{j,\ell,k} \langle u, \boldsymbol{\psi}_{j,\ell,k} \rangle \overline{\boldsymbol{\psi}_{j,\ell,k}}$. Note that the scale and wedge depending sampling destroys the translation invariance and the improved denoising property of TI systems [8, 16].

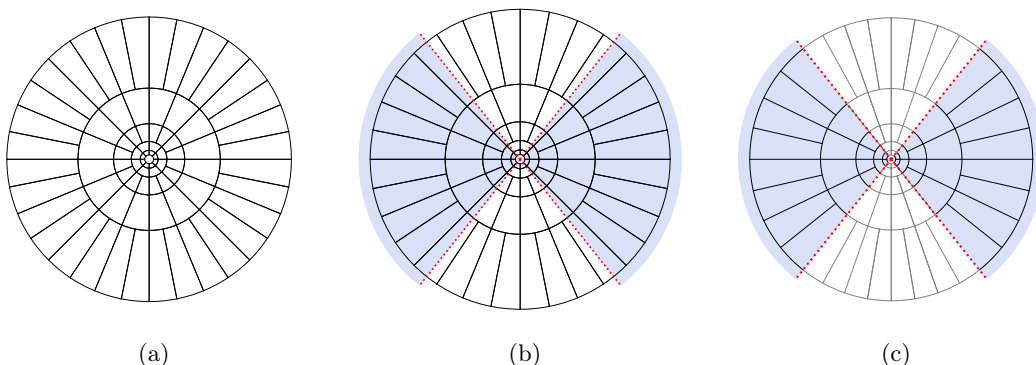


Figure A.1: (a) Standard curvelet tiling. (b) Visible wedge W_I indicated in blue and non-adapted standard curvelet tiling. (c) Visible wedge W_I and wedge adapted tiling.

A.2 Wedge adaption

Due to the limited angular range, the essential support of the Fourier transformed curvelets near the boundary of the visible wedge W_I is not fully contained in W_I ; see Figure A.1b. This results in an associated curvelet transform that is not well adapted to the kernel of the limited Radon transform [12]. In order to adapt to the visible wedge we modify the standard angular tiling and define two systems $(\boldsymbol{\psi}_{j,\ell}^{\text{vis}})_{j,\ell}$ and $(\boldsymbol{\psi}_{j,\ell}^{\text{inv}})_{j,\ell}$ that we call the visible and invisible parts of the curvelet family. For that purpose we define the adjusted angular windows $V^{\text{vis}}(\phi)$ and V^{inv} and make sure that the windows at the boundary sum up to one.

Now the wedge-adapted TI curvelets $\psi_{j,\ell}^{\text{vis}}$, $\psi_{j,\ell}^{\text{inv}}$ are defined as in (A.1) with V replaced by V^{vis} , V^{inv} respectively. As in Theorem A.1 one shows that the family $(\psi_{j,\ell}^{\text{vis}}, \psi_{j,\ell}^{\text{inv}})_{j,\ell}$ forms a TI frame of $L^2(\mathbb{R}^2)$. Opposed to the standard TI curvelet frame $(\psi_{j,\ell})_{j,\ell}$ it has controlled overlap at the boundary between visible and invisible frequencies. In a similar manner we could construct wedge adapted curvelets where we use different numbers N_j^d for each of the four basic wedges. Finally, note that each of windows has finite bandwidth. Thus similar to the case of the standard curvelets we can use Shannon sampling theorem define a wedge adapted curvelet frame by wedge adapted sampling. A detailed mathematical analysis of properties of its properties is beyond the scope of this paper.

References

- [1] R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*, 10(6):1217, 1994.
- [2] L. Borg, J. S. Jørgensen, J. Friel, and E. T. Quinto. Analyzing reconstruction artifacts from arbitrary incomplete x-ray ct data. *SIAM J. Imaging Sci.*, 11(4):2786–2814, 2018.
- [3] T. A. Bubba, D. Labate, G. Zanghirati, and S. Bonettini. Shearlet-based regularized reconstruction in region-of-interest computed tomography. *Math. Model. Nat. Phenom.*, 13(4):34, 2018.
- [4] E. J. Candes and D. L. Donoho. Curvelets and reconstruction of images from noisy radon data. In *Wavelet applications in signal and image processing VIII*, volume 4119, pages 108–117. SPIE, 2000.
- [5] E. J. Candes and D. L. Donoho. Recovering edges in ill-posed inverse problems: Optimality of curvelet frames. *Ann. Stat.*, 30(3):784–842, 2002.
- [6] E. J. Candes and F. Guo. New multiscale transforms, minimum total variation synthesis: Applications to edge-preserving image reconstruction. *Signal Processing*, 82(11):1519–1543, 2002.
- [7] A. Chambolle and T. Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *J. Math. Imaging Vision*, 40(1):120–145, 2011.
- [8] R. R. Coifman and D. L. Donoho. Translation-invariant de-noising. In *Wavelets and statistics*, pages 125–150. Springer, 1995.
- [9] P. L. Combettes and J.-C. Pesquet. Proximal splitting methods in signal processing. *Fixed-point algorithms for inverse problems in science and engineering*, pages 185–212, 2011.
- [10] I. Daubechies. *Ten lectures on wavelets*. SIAM, 1992.

- [11] A. Ebner, J. Friel, D. Lorenz, J. Schwab, and M. Haltmeier. Regularization of inverse problems by filtered diagonal frame decomposition. *Appl. Comput. Harmon. Anal.*, 62:66–83, 2023.
- [12] J. Friel. Sparse regularization in limited angle tomography. *Appl. Comput. Harmon. Anal.*, 34(1):117–141, 2013.
- [13] J. Friel and M. Haltmeier. Sparse regularization of inverse problems by operator-adapted frame thresholding. In *Mathematics of Wave Phenomena*, pages 163–178. Springer, 2020.
- [14] J. Friel and E. T. Quinto. Characterization and reduction of artifacts in limited angle tomography. *Inverse Problems*, 29(12):125007, 2013.
- [15] J. Friel and E. T. Quinto. Limited data problems for the generalized radon transform in \mathbb{R}^n . *SIAM J. Math. Anal.*, 48(4):2301–2318, 2016.
- [16] S. Göppel, J. Friel, and M. Haltmeier. Translation invariant diagonal frame decomposition of inverse problems and their regularization. *Inverse Problems*, 2022.
- [17] M. Grasmair, M. Haltmeier, and O. Scherzer. Sparse regularization with ℓ^q penalty term. *Inverse Problems*, 24(5):055020, 2008.
- [18] M. Haltmeier, H. Li, and A. Munk. A variational view on statistical multiscale estimation. *Annu. Rev. Stat. Appl.*, 9:343–372, 2022.
- [19] C. Kai, J. Min, Z. Qu, J. Yu, and S. Yi. Moreau-envelope-enhanced nonlocal shearlet transform and total variation for sparse-view ct reconstruction. *Meas. Sci. Technol.*, 32(1):015405, 2020.
- [20] G. Kutyniok and W.-Q. Lim. Compactly supported shearlets are optimally sparse. *J. Approx. Theory*, 163(11):1564–1589, 2011.
- [21] X. Luo, W. Yu, and C. Wang. An image reconstruction method based on total variation and wavelet tight frame for limited-angle ct. *IEEE Access*, 6:1461–1470, 2017.
- [22] S. Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, Inc., USA, 3rd edition, 2008.
- [23] F. Natterer. *The mathematics of computerized tomography*. SIAM, 2001.
- [24] M. Persson, D. Bone, and H. Elmqvist. Total variation norm for three-dimensional iterative reconstruction in limited view angle tomography. *Phys. Med. Biol.*, 46(3):853, 2001.
- [25] E. T. Quinto. Singularities of the X-ray transform and limited data tomography in \mathbb{R}^2 and \mathbb{R}^3 . *SIAM J. Math. Anal.*, 24(5):1215–1225, 1993.

- [26] E. T. Quinto. Artifacts and visible singularities in limited data x-ray tomography. *Sens. Imaging*, 18(1):1–14, 2017.
- [27] M. Rantala, S. Vanska, S. Jarvenpaa, M. Kalke, M. Lassas, J. Moberg, and S. Siltanen. Wavelet-based reconstruction for limited-angle x-ray tomography. *IEEE Trans. Med. Imaging*, 25(2):210–217, 2006.
- [28] B. Sahiner and A. E. Yagle. Limited angle tomography using wavelets. In *Nuclear Science Symposium and Medical Imaging Conference*, pages 1912–1916. IEEE, 1993.
- [29] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen. *Variational methods in imaging*. Springer, New York, 2009.
- [30] W. P. Segars, M. Mahesh, T. J. Beck, E. C. Frey, and B. M. Tsui. Realistic CT simulation using the 4D XCAT phantom. *Med. Phys.*, 35(8):3800–3808, 2008.
- [31] E. Y. Sidky and X. Pan. Image reconstruction in circular cone-beam computed tomography by constrained, total-variation minimization. *Phys. Med. Biol.*, 53(17):4777, 2008.
- [32] K. T. Smith, D. C. Solmon, and S. L. Wagner. Practical and mathematical aspects of the problem of reconstructing objects from radiographs. *Bull. Amer. Math. Soc.*, 83(6):1227–1270, 1977.
- [33] B. Vandeghinste, B. Goossens, R. Van Holen, C. Vanhove, A. Pizurica, S. Vandenberghe, and S. Staelens. Combined shearlet and TV regularization in sparse-view CT reconstruction. In *2nd International Meeting on image formation in X-ray Computed Tomography*, 2012.
- [34] B. Vandeghinste, B. Goossens, R. Van Holen, C. Vanhove, A. Pižurica, S. Vandenberghe, and S. Staelens. Iterative CT reconstruction using shearlet-based regularization. *IEEE Trans. Nucl. Sci.*, 60(5):3305–3317, 2013.
- [35] J. Velikina, S. Leng, and G.-H. Chen. Limited view angle tomographic image reconstruction via total variation minimization. In *Medical Imaging 2007*, volume 6510, pages 709–720. SPIE, 2007.
- [36] T. Wang, K. Nakamoto, H. Zhang, and H. Liu. Reweighted anisotropic total variation minimization for limited-angle ct reconstruction. *IEEE Trans. Nucl. Sci.*, 64(10):2742–2760, 2017.