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Translation invariant diagonal frame decomposition of inverse problems and their regularization

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Abstract

Solving inverse problems is central to a variety of important applications, such as biomedical image reconstruction and non-destructive testing. These problems are characterized by the sensitivity of direct solution methods with respect to data perturbations. To stabilize the reconstruction process, regularization methods have to be employed. Well-known regularization methods are based on frame expansions, such as the wavelet-vaguelette (WVD) decomposition, which are well adapted to the underlying signal class and the forward model and allow efficient implementation. However, it is well known that the lack of translational invariance of wavelets and related systems leads to specific artifacts in the reconstruction. To overcome this problem, in this paper we introduce and analyze the concept of translation invariant diagonal frame decomposition (TI-DFD) of linear operators. We prove that a TI-DFD combined with a regularizing filter leads to a convergent regularization method with optimal convergence rates. As illustrative example, we construct a wavelet-based TI-DFD for one-dimensional integration, where we also investigate our approach numerically. The results indicate that filtered TI-DFDs eliminate the typical wavelet artifacts when using standard wavelets and provide a fast, accurate, and stable solution scheme for inverse problems.

Keywords. Inverse problems, regularization, convergence rates, translation invariance, frames, wavelets, vaguelettes, operator decomposition

1 Introduction

In this paper, we study with the stable solution of linear inverse problems. Such problems aim at recovering an unknown image or function $f_\star \in L^2(\mathbb{R}^d)$ from data

$$g^\delta = \mathcal{K} f_\star + \eta \,. \tag{1.1}$$

Here $\mathcal{K}\colon \operatorname{dom}(\mathcal{K}) \subseteq L^2(\mathbb{R}^d) \to \mathbb{Y}$ is a closed not necessarily bounded, linear operator between $L^2(\mathbb{R}^d)$ and another Hilbert space $\mathbb{Y}$ and η denotes the unknown data distortion, which in our analysis is assumed to satisfy $\|\eta\| \leq \delta$ for the noise level $\delta > 0$. Inverse problems of the form (1.1) are often ill-posed, meaning that the solution is either not well-defined or unstable with respect to data perturbations. In order to stabilize the solution process one has to apply regularization methods [9, 20]. The basic idea of regularization is to incorporate prior information and to relax the exact solution concept to make the inversion process stable.

A common and successful approach for solving inverse problems is variational regularization. In that context, prior information is incorporated in the form of a regularizer $\mathrm{R}\colon L^2(\mathbb{R}^d) \to [0,\infty]$ acting as penalty and regularized solutions are constructed as minimizers of the generalized Tikhonov functional

$$\mathrm{T}_{\alpha, g^\delta}(f) \coloneqq \|\mathcal{K} f - g^\delta\|^2 + \alpha \mathrm{R}(f) \,. \tag{1.2}$$

A particularly important and well-studied special case is classical Tikhonov regularization where $\mathrm{R}(f) = \|f\|^2$. In this case, assuming a singular value decomposition (SVD) $(u_\lambda, v_\lambda, \sigma)_{\lambda\in\Lambda}$ of the operator $\mathcal{K}$, the minimizer of the Tikhonov functional can be explicitly computed as $f_\alpha^\delta = \sum_{\lambda\in\Lambda} (\sigma/(\sigma^2+\alpha)) \cdot \langle g^\delta, v_\lambda\rangle \, u_\lambda$. Opposed to general variational regularization, where the Tikhonov functional (1.2) has to be minimized via iterative methods, the SVD based reconstruction can be evaluated directly and efficiently. The same holds true for more general filter based methods where the Tikhonov filter function $\Phi_\alpha(\sigma) = \sigma/(\sigma^2+\alpha)$ is replaced by general functions. Another typical choice in that context is the truncation filter $\Phi_\alpha(\sigma) = \chi_{[\alpha,\infty)}(\sigma^2)$ which leads to the also well-known truncated SVD reconstruction method.

1.1 Frame-based regularization

Although the SVD is a very successful tool to design stable inversion schemes for (1.1), it comes with several shortcomings. Firstly, given a particular linear operator, the SVD may not be known analytically and hard to be computed numerically. Furthermore, the underlying orthonormal basis elements u_n, v_n of the singular system are derived only from the operator $\mathcal{K}$ and are not adapted to signal classes of interest. On the other hand it is well known that different function systems such as wavelet frames are better adapted to functions classes of practical relevance. Therefore researchers studied frame based regularization methods where the regularizer in (1.2) has the form $\mathrm{R}(f) = \sum_{\lambda \in \Lambda} r_\lambda(\langle u_\lambda, f \rangle)$ for some frame or basis $(u_\lambda)_{\lambda \in \Lambda}$ of $L^2(\mathbb{R}^d)$ and univariate functionals r_λ. Such methods have been successful applied and analyzed in various settings [5, 6, 11, 15, 18, 19, 20].

Variational frame based methods have the drawback that they again require computationally costly iterative solution methods. To overcome this issue, diagonal frame decompositions (DFD) have been developed as an alternative tool. A DFD for an operator $\mathcal{K}$ is given as a family $(u_\lambda, v_\lambda, \kappa_\lambda)_{\lambda \in \Lambda}$, where $(u_\lambda)_{\lambda \in \Lambda}$ and $(v_\lambda)_{\lambda \in \Lambda}$ are chosen as frames instead of orthonormal bases as it is in the case of an SVD. Additionally, the κ_λ are generalized singular values or so-called quasi singular values which satisfy $\mathcal{K}^* v_\lambda = \kappa_\lambda u_\lambda$. As proposed and analyzed in [8] such DFDs yield explicit regularization methods in the form of filtered DFDs

$$f_\alpha^\delta = \sum_{\lambda \in \Lambda} \Phi_\alpha(\kappa_\lambda) \, \langle g^\delta, v_\lambda \rangle w_\lambda \,, \tag{1.3}$$

where $(\Phi_\alpha)_{\alpha > 0}$ is a regularizing filter and $(w_\lambda)_{\lambda \in \Lambda}$ a dual frame to $(u_\lambda)_{\lambda \in \Lambda}$. Such filtered DFDs combine advantages of filter based and frame based methods. It may be seen as a generalization of the filtered SVD using redundant frames instead orthogonal bases. Precise definitions of DFDs and regularizing filters are provided in Definitions 2.6 and 3.1 below. DFDs also generalize the wavelet vaguelette decomposition (WVD) introduced in the seminal paper [7]; for related constructions using curvelet and shearlet frames see [1, 3]. DFDs for general frames has bee first introduced in [10] where they are analyzed in combination with soft thresholding. A convergence analysis of filtered DFDs for general linear filters has first been provided in [8] and later in [13].

1.2 Proposed translation invariant DFD

Classical frame based reconstruction approaches such as the WVD lack translation invariance of the underlying frame which introduces well-known wavelet artifacts in the reconstruction. Translation invariant wavelet frames are known to perform better in that regard for simple tasks such as denoising [2, 17]. Thus, it is natural to introduce translation invariant frames for general inverse problems as we will do in the present paper. Furthermore, we present a complete convergence analysis for the corresponding filtered translation invariant frame decomposition (TI-DFD).

A translation invariant frame of $L^2(\mathbb{R})$, which is not a frame in the classical sense, consists of a family $(u_\lambda)_{\lambda \in \Lambda} \in L^2(\mathbb{R}^2)^\Lambda$ such that any $f \in L^2(\mathbb{R}^d)$ can be stably analyzed and reconstructed via convolutions with u_λ. In this paper we introduce the TI-DFD of a linear operator which as family $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda \in \Lambda}$ where $(u_\lambda)_{\lambda \in \Lambda} \in L^2(\mathbb{R}^2)$ is TI-frame, $\mathcal{V}_\lambda^* \colon \mathbb{Y} \to L^2(\mathbb{R}^d)$ are suitable linear operators and $\kappa_\lambda > 0$ are generalized singular values with $\mathcal{V}_\lambda^*(\mathcal{K} f) = \kappa_\lambda \cdot (u_\lambda^* * f)$. Given a regularizing filter

$(\Phi_\alpha)_{\alpha>0}$ and a dual TI-frame $(w_\lambda)_{\lambda\in\Lambda} \in L^2(\mathbb{R}^2)$ we will demonstrate that

$$f_\alpha^\delta = \sum_{\lambda\in\Lambda} \Phi_\alpha(\kappa_\lambda)\,(w_\lambda * (\mathcal{V}_\lambda^* g^\delta)) \tag{1.4}$$

is a regularization method and we additionally derive convergence rates. Precise definitions of TI frames and TI-DFDs are given in Definitions 2.1, 2.8 below. For illustrative purpose, will construct an example of a TI-DFD for the 1D integration operator using the TI wavelet transform. We will demonstrate that the resulting filtered TI-WVD works well and overcomes wavelet reconstruction artifacts present in the standard WVD.

1.3 Outline

The remainder of this paper is organized as follows. In Section 2 we introduce the concept of TI-DFD and provide some of its properties. In Section 3 we introduce and analyze the proposed regularization concept in the form of the filtered TI-DFD. We prove regularization properties and derive order optimal convergence rates. In Section 4 we construct a TI-DFD for the 1D integral operator for which we support our theory by numerical simulation. The paper concludes with a short summary and outlook given in Section 5.

2 TI-DFD of linear operators

Throughout this paper, let $\mathcal{K}\colon \operatorname{dom}(\mathcal{K}) \subseteq L^2(\mathbb{R}^d) \to \mathbb{Y}$ be a linear, closed and not necessarily bounded operator between $L^2(\mathbb{R}^d)$ and another Hilbert space $\mathbb{Y}$.

We write $u^*(x) := \overline{u(-x)}$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. The Fourier transform of $f \in L^2(\mathbb{R}^d)$ is denoted by $\hat{f} = \mathcal{F}f$ where $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\langle \xi, x\rangle}\,\mathrm{d}x$ for integrable functions. Recall that the Fourier transform turns convolution into multiplication. In particular for $u, f \in L^2(\mathbb{R}^d)$ with $\hat{u} \in L^\infty(\mathbb{R}^d)$, the convolution $u^* * f \in L^2(\mathbb{R}^d)$ is well defined and $u^* * f = \mathcal{F}^{-1}((\overline{\mathcal{F}u}) \cdot (\mathcal{F}f))$.

2.1 TI-frames

In this subsection we recall the concept of TI-frames and collect properties we will require for our purpose. Related issues can found in [16, Sec. 5.2].

Definition 2.1 (TI-frame)**.** *Let Λ be an at most countable index set. A family $(u_\lambda)_{\lambda\in\Lambda} \in L^2(\mathbb{R}^d)^\Lambda$ is called a translation invariant frame (TI-frame) for $L^2(\mathbb{R}^d)$ if $\hat{u}_\lambda \in L^\infty(\mathbb{R}^d)$ for all $\lambda \in \Lambda$ and there exist constants $A, B > 0$ such that*

$$\forall f \in L^2(\mathbb{R}^d)\colon \quad A\|f\|^2 \le \sum_{\lambda\in\Lambda} \|u_\lambda^* * f\|^2 \le B\|f\|^2\,. \tag{2.1}$$

A TI-frame $(u_\lambda)_{\lambda\in\Lambda}$ is called tight if (2.1) *holds with TI-frame founds $A = B = 1$.*

Because $\hat{u}_\lambda \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$, we have $u_\lambda^* * f = \mathcal{F}^{-1}((\overline{\mathcal{F}u_\lambda}) \cdot (\mathcal{F}f)) \in L^2(\mathbb{R}^d)$. From Plancherel's theorem we get $\|u_\lambda^* * f\|^2 = (2\pi)^{-1} \int_{\mathbb{R}^d} |\mathcal{F}u_\lambda|^2 |\mathcal{F}f|^2$. The right inequality in the TI-frame property (2.1) therefore in particular implies $(u_\lambda^* * f)_{\lambda\in\Lambda} \in \ell^2(\Lambda, L^2(\mathbb{R}^d))$. Here and below for a Hilbert space $\mathbb{H}$ we write $\ell^2(\Lambda, \mathbb{H})$ for the Hilbert space of all $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in \mathbb{H}^\Lambda$ satisfying $\|\mathbf{c}\|_\Lambda^2 \coloneqq \sum_{\lambda\in\Lambda} \|c_\lambda\|^2 < \infty$ with inner product $\langle \mathbf{c}, \mathbf{d} \rangle_\Lambda \coloneqq \sum_{\lambda\in\Lambda} \langle c_\lambda, d_\lambda \rangle$.

Along with the definition of a TI-frame, it is convenient to introduce the following TI versions of synthesis, analysis and frame operators.

Definition 2.2 (TI-synthesis, TI-analysis and TI-frame operator)**.** *Let $(u_\lambda)_{\lambda\in\Lambda}$ be a TI-frame.*

(a) $\mathcal{U} \colon \ell^2(\Lambda, L^2(\mathbb{R}^d)) \to L^2(\mathbb{R}^d) \colon (c_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} u_\lambda * c_\lambda$ *is called TI-synthesis operator.*

(b) $\mathcal{U}^* \colon L^2(\mathbb{R}^d) \to \ell^2(\Lambda, L^2(\mathbb{R}^d)) \colon f \mapsto (u_\lambda^* * f)_{\lambda\in\Lambda}$ *is called TI-analysis operator.*

(c) $\mathcal{U}\mathcal{U}^* \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is called TI-frame operator.*

Note the TI-analysis operator is the adjoint of the TI-synthesis operator. Using the definition of the TI-analysis operator and the norm on $\ell^2(\Lambda, L^2(\mathbb{R}^d))$, we can rewrite the frame condition (2.1) as $\forall f \in L^2(\mathbb{R}^d) \colon A\|f\| \leq \|\mathcal{U}^* f\|_\Lambda^2 \leq B\|f\|$. The right inequality in (2.1) states that the TI-analysis operator $\mathcal{U}^*$ is a well-defined bounded linear operator, and the left inequality states that $\mathcal{U}^*$ is bounded from below. In particular, its Moore-Penrose inverse $(\mathcal{U}^*)^\ddagger \colon \ell^2(\Lambda, L^2(\mathbb{R}^d)) \to L^2(\mathbb{R}^d)$ is bounded and given by $(\mathcal{U}^*)^\ddagger = (\mathcal{U}\mathcal{U}^*)^{-1}\mathcal{U}$. Finally, $A = \|(\mathcal{U}^*)^\ddagger\|^{-2}$ and $B = \|\mathcal{U}^*\|^2$ are the optimal TI-frame bounds for (2.1).

Let us collect some further useful properties.

Proposition 2.3 (Properties of TI-frames)**.** *Let $(u_\lambda)_{\lambda\in\Lambda} \in L^2(\mathbb{R}^d)^\Lambda$.*

(a) $(u_\lambda)_{\lambda\in\Lambda}$ *TI-frame with bounds* $A, B > 0 \iff 2\pi A \leq \sum_{\lambda\in\Lambda} |\hat{u}_\lambda|^2 \leq 2\pi B$.

(b) $(u_\lambda)_{\lambda\in\Lambda}$ *tight TI-frame* $\iff \sum_{\lambda\in\Lambda} |\hat{u}_\lambda|^2 = 2\pi$.

(c) $(u_\lambda)_{\lambda\in\Lambda}$ *TI-frame* $\Rightarrow (u_\mu^\ddagger)_{\mu\in\Lambda}$ *with* $u_\mu^\ddagger \coloneqq \mathcal{F}^{-1}(2\pi \hat{u}_\mu / \sum_{\lambda\in\Lambda} |\hat{u}_\lambda|^2)$ *is a TI-frame.*

(d) $(\mathcal{U}^*)^\ddagger$ *is the synthesis operator of* $(u_\lambda^\ddagger)_{\lambda\in\Lambda}$.

(e) $(u_\lambda)_{\lambda\in\Lambda}$ *TI-frame* $\Rightarrow \mathrm{Id} = (\mathcal{U}^*)^\ddagger \mathcal{U}^*$.

(f) $(u_\lambda)_{\lambda\in\Lambda}$ *tight TI-frame* $\Rightarrow \mathrm{Id} = \mathcal{U}\mathcal{U}^*$.

Proof. Items (a), (b) follow from (2.1) and the Plancherel theorem. Item (c), (d) follow from (a) and straight forward computations. Items (e), (f) hold because the Moore-Penrose inverse is a right inverse and $(\mathcal{U}^*)^\ddagger = \mathcal{U}$ for TI-tight frames. □

Definition 2.4 (Dual TI-frame). *Let $(u_\lambda)_{\lambda\in\Lambda}$, $(w_\lambda)_{\lambda\in\Lambda}$ be TI-frames with TI-sythesis operators $\mathcal{U}$ and $\mathcal{W}$. We call $(w_\lambda)_{\lambda\in\Lambda}$ a dual TI-frame to $(u_\lambda)_{\lambda\in\Lambda}$ if $\mathcal{W}\mathcal{U}^* = \mathrm{Id}$.*

On easily verifies that dual frame condition is equivalent to the identity $\sum_\lambda(\mathcal{F}w_\lambda)\cdot(\overline{\mathcal{F}u_\lambda}) = 2\pi$. Moreover, in this case,

$$\forall f \in L^2(\mathbb{R}^d)\colon \quad f = \mathcal{W}\mathcal{U}^* f = \sum_{\lambda\in\Lambda} w_\lambda * (u_\lambda^* * f)\,. \tag{2.2}$$

The reproducing formula (2.2) in particular holds for $(u_\lambda^\ddagger)_{\lambda\in\Lambda} = (w_\lambda)_{\lambda\in\Lambda}$, which is referred to as the canonical dual TI-frame. According to the characterization via the Moore-Penrose inverse, the canonical dual applied to coefficients $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda, L^2(\mathbb{R}^d))$ is characterized as minimizer of the least square functional $f \mapsto \|\mathcal{U}^* f - \mathbf{c}\|^2$. In some applications other left inverses and dual TI-frames different may be of interest.

We conclude this subsection by drawing some connections to classical frames. This will be part of our reasoning for introducing TI-DFDs as additional tool besides the DFD and SVD.

Remark 2.5 (Frames versus TI-frames). *We again point out that a TI-frame is not a frame in the classical sense. A family $(u_{\lambda,k})_{(\lambda,k)\in\Lambda\times\mathbb{Z}^d} \in L^2(\mathbb{R}^d)^{\Lambda\times\mathbb{Z}^n}$ for some parameter $n\in\mathbb{N}$ is a frame of $L^2(\mathbb{R}^d)$ (in the classical sense) if there exist frame bounds $A, B \in (0,\infty)$ such that*

$$\forall f \in L^2(\mathbb{R}^d)\colon \quad A\|f\|^2 \le \sum_{\lambda\in\Lambda}\sum_{k\in\mathbb{Z}^n} |\langle f, u_{\lambda,k}\rangle|^2 \le B\|f\|^2\,. \tag{2.3}$$

The identity $u_\lambda^* * f(x) = \langle f, u_\lambda((\,\cdot\,)-x)\rangle$ shows that a TI-frame may be seen as a generalized notion of a frame using the semi-discrete index set $\Lambda\times\mathbb{R}^d$, frame elements $u_{\lambda,x} = u_\lambda((\,\cdot\,)-x)$ and the squared L^2-norm replacing the inner ℓ^2-norm in (2.3).

Conversely, classical frames can be obtained from TI-frames by discretizing the convolutions $u_\lambda^* * f$. As an example in the context of wavelet analysis, consider a mother wavelet $u \in L^2(\mathbb{R})$ and set $u_j(x) := 2^j u(2^j x)$. The family $(u_j)_{j\in\mathbb{Z}}$ is a TI-frame if $A \le \sum_{j\in\mathbb{Z}}|\hat{u}(2^{-j}\omega)|^2 \le B$. Under additional assumptions [4] the family $(u_{j,k})_{j,k\in\mathbb{Z}}$ defined by $u_{j,k}(x) = 2^{-j/2}u_j(x-2^jk)$ becomes a frame of $L^2(\mathbb{R}^d)$. Note that the sampling step size 2^j for constructing a frame depends on the scale j, which in particular destroys the translation invariance of the underlying TI-frame.

2.2 Review: diagonal frame decomposition

Before actually introducing TI-DFD in the next subsection, we first review the WVD and more general DFDs of linear operators. The WVD was introduced in [7] for several integral operators. The more general notation of a DFD used below is taken from [8]. Note that we formulate the following definition for an index set of the form $\Lambda\times\mathbb{Z}^n$ to make the comparison with TI frames more obvious and to be closer to the notion of wavelet frames.

Definition 2.6 (DFD)**.** *The system* $(u_{\lambda,k}, v_{\lambda,k}, \kappa_{\lambda,k})_{(\lambda,k)\in\Lambda\times\mathbb{Z}^n}$*, for* $n \in \mathbb{N}$ *and countable* Λ*, is called DFD for* $\mathcal{K}$*, if the following properties hold:*

(DFD1) $(u_{\lambda,k})_{\lambda,k\in\Lambda\times\mathbb{Z}^d}$ *is a frame of* $L^2(\mathbb{R}^d)$*.*

(DFD2) $(v_{\lambda,k})_{\lambda,k\in\Lambda\times\mathbb{Z}^d}$ *is a frame of* $\overline{\operatorname{ran}\mathcal{K}}$*.*

(DFD3) $\forall(\lambda,k) \in \Lambda\times\mathbb{Z}^n \colon \mathcal{K}^* v_{\lambda,k} = \kappa_\lambda u_{\lambda,k}$*.*

If $(u_{j,k})_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d}$ is a wavelet frame we refer to the DFD as WVD. The elements $v_{\lambda,k}$ in this case are called vaguelettes. With a dual frame $(w_{\lambda,k})_{(\lambda,k)\in\Lambda\times\mathbb{Z}^d}$, the following reproducing formula holds for all $f \in \operatorname{dom}(\mathcal{K})$:

$$f = \sum_{\lambda\in\Lambda} \frac{1}{\kappa_\lambda} \sum_{k\in\mathbb{Z}^n} \langle \mathcal{K} f, v_{\lambda,k}\rangle w_{\lambda,k}\,. \tag{2.4}$$

The DFD includes the SVD, in which case $n = 0$ and $(u_{\lambda,0})_{\lambda\in\Lambda}$ and $(v_{\lambda,0})_{\lambda\in\Lambda}$ are orthonormal bases, and the WVD where $n = d$ and $(u_{\lambda,k})_{(\lambda,k)\in\Lambda\times\mathbb{Z}^d}$ is a wavelet frame. As mentioned in the introduction the WVD can be better adapted to the signal class compared to the SVD.

In [7] the WVD has been derived for various examples including the Radon transform and 1D integration. Building upon the WVD concept, several related constructions have been studied. For example, biorthogonal curvelet and shearlet decompositions for the Radon transform are established in [1, 3]. The general concept of DFDs has been introduced in [10] and in [8] a filter based regularization theory based on the DFD has been developed.

Wavelet frames suffer from specific artifacts after coefficient filtering, mainly due to the lack of translation invariance. Therefore we will develop a related concept using TI-frames instead of frames. In order to further motivate our approach, below we give a representation of the WVD using sampled convolutions and point out where TI invariance can be restored.

Remark 2.7 (WVD in sampled convolution form)**.** *Consider a 1D orthonormal wavelet basis* $(u_{j,k})_{j,k\in\mathbb{Z}}$ *of* $L^2(\mathbb{R})$ *with* $u_{j,k}(x) \coloneqq 2^{-j/2}u_j(x - 2^j k)$ *and* $u_j(x) \coloneqq 2^j u(2^j x)$ *with mother wavelet* $u \in L^2(\mathbb{R})$*. Moreover, let* $(u_{j,k}, v_{j,k}, \kappa_j)_{j,k\in\mathbb{Z}}$ *be the corresponding WVD for* $\mathcal{K}$*. For* $f \in L^2(\mathbb{R}^d)$ *and* $\mathbf{a} = (a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ *define the downsampled and upsampled convolutions* $(u_j^* \circledast_j f)_k \coloneqq 2^{-j/2} \cdot (u_j^* * f)(2^j k)$ *and* $(u_j \circledast_j \mathbf{a})(x) \coloneqq 2^{-j/2} \sum_{k\in\mathbb{Z}} a_k u_j(x - k2^j)$ *respectively. The WVD reconstruction formula can be written as*

$$f = \sum_{j\in\mathbb{Z}} u_j \circledast_j \left(u_j^* \circledast_j f\right), \tag{2.5}$$

$$u_j^* \circledast_j f = \frac{1}{\kappa_j}(\langle \mathcal{K} f, v_{j,k}\rangle)_{k\in\mathbb{Z}}\,. \tag{2.6}$$

Here $u_j^* \circledast_j f = 2^{-j/2}\,(u_j^* * f)(2^j k)$ *are samples of the continuous convolution and the level-depending sampling destroys the translational invariance. Note that* (2.5), (2.6) *is equivalent to* (2.4)*. The proposal of this paper is can be seen as a way to restore translational invariance by replacing the sampled convolutions in* (2.5) *by the non-sampled counterparts and to modify* (2.6) *accordingly.*

2.3 Introducing the TI-DFD

We now introduce the TI-DFD as the central concept of this paper. We denote by $B(\mathbb{Y}, L^2(\mathbb{R}^d))$ the space of all bounded linear operators from $\mathbb{Y}$ to $L^2(\mathbb{R}^d)$.

Definition 2.8 (TI-DFD)**.** *We call the system $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda\in\Lambda}$ a translation invariant frame decomposition (TI-DFD) for $\mathcal{K}$, if the following properties hold:*

(TI1) $(u_\lambda)_{\lambda\in\Lambda} \in L^2(\mathbb{R}^d)^\Lambda$ *is a TI-frame for* $L^2(\mathbb{R}^d)$.

(TI2) $\forall\lambda \in \Lambda$ *we have* $\mathcal{V}_\lambda^* \in B(\mathbb{Y}, L^2(\mathbb{R}^d))$ *and* $\forall g \in \overline{\operatorname{ran}\mathcal{K}}\colon \sum_{\lambda\in\Lambda}\|\mathcal{V}_\lambda^* g\|^2 \asymp \|g\|^2$.

(TI3) $\forall\lambda \in \Lambda\colon \kappa_\lambda \in (0,\infty)$ *and* $\forall f \in \operatorname{dom}(\mathcal{K})\colon \mathcal{V}_\lambda^*(\mathcal{K}f) = \kappa_\lambda\,(u_\lambda^* * f)$.

Let us compare a TI-DFD to a regular DFD $(u_{\lambda,k}, v_{\lambda,k}, \kappa_\lambda)_{(\lambda,k)\in\Lambda\times\mathbb{Z}^d}$, where $u_{\lambda,k}(x) = u_\lambda(x - M_\lambda k)$ with sampling matrix $M_\lambda \in \mathbb{R}^{d\times d}$. Among others, such forms includes WVDs and DFDs with curvelet or shearlet frames. For example, for the 1D wavelet transform (Remark 2.7) we have $\Lambda = \mathbb{Z}$, and for the 2D wavelet transform we have $\Lambda = \mathbb{Z}\times\{\mathrm{H},\mathrm{V},\mathrm{D}\}$ representing horizontal (H), vertical (V) and diagonal (D) wavelets at scale $j \in \mathbb{Z}$. In such situations, (TI1), (TI2) replace the frame conditions (DFD1), (DFD2). In item (TI3), $u_\lambda^* * f$ takes over the role of the inner products $\langle u_{\lambda,k}, f\rangle = (u_\lambda^* * f)(M_\lambda k)$ and $\mathcal{V}_\lambda^* g$ takes over the role of $(\langle v_{\lambda,k}, g\rangle)_{k\in\mathbb{Z}^d}$. Finally, the identity $\mathcal{V}_\lambda^*\mathcal{K}f = \kappa_\lambda\, u_\lambda^* * f$ replaces the quasi-singular value relation $\langle v_{\lambda,k}, \mathcal{K}f\rangle = \kappa_\lambda\langle u_{\lambda,k}, f\rangle$. In particular, the standard DFD may be seen as kind of discretization of the TI-frame decomposition, where the discretization depends on the index λ.

Remark 2.9 (Operator $\mathcal{V}^*$)**.** *According to (TI2) there exist constants $A, B \in (0,\infty)$ such that $A\|g\|^2 \le \sum_{\lambda\in\Lambda}\|\mathcal{V}_\lambda^* g\|^2 \le B\|g\|^2$ for all $g \in \overline{\operatorname{ran}\mathcal{K}}$. Equivalently, the operator*

$$\mathcal{V}^*\colon \mathbb{Y} \to \ell^2(\Lambda, L^2(\mathbb{R}^d))\colon f \mapsto (\mathcal{V}_\lambda^* f)_{\lambda\in\Lambda} \tag{2.7}$$

is well defined, bounded and bounded from below with norm bounds $\|\mathcal{V}\| \le B^{1/2}$ and $\|(\mathcal{V}^*)^\ddagger\| \le A^{-1/2}$. It plays plays a similar role in $\mathbb{Y}$ as the TI-analysis operator $\mathcal{U}^*$ does in $L^2(\mathbb{R}^d)$. However, operator $\mathcal{V}^*$ is not of the convolution form in general.

Similar to the standard DFD we have the following TI-DFD reconstruction formula.

Proposition 2.10 (Exact reconstruction formula via TI-DFD)**.** *Let $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda\in\Lambda}$ be a TI-DFD for $\mathcal{K}$ and $(w_\lambda)_{\lambda\in\Lambda}$ be a dual TI-frame for $(u_\lambda)_{\lambda\in\Lambda}$. Then*

$$\forall g \in \operatorname{ran}(\mathcal{K})\colon \quad \mathcal{K}^{-1}g = \sum_{\lambda\in\Lambda} w_\lambda * (\kappa_\lambda^{-1}\cdot(\mathcal{V}_\lambda^* g)) = \mathcal{W}\big((\kappa_\lambda^{-1}\mathcal{V}_\lambda^* g)_{\lambda\in\Lambda}\big)\,. \tag{2.8}$$

Proof. With $g = \mathcal{K}f$, identity (2.8) follows after inserting (TI3) into (2.2). □

Similar to the SVD and the DFD reconstruction formulas, (2.8) reflects the ill-posedness of $\mathcal{K}^{-1}$ in terms of the quasi-singular values that are potentially accumulating at zero. More precisely, we have the following results.

Theorem 2.11 (Characterization of ill-posedness via TI-DFD)**.** *Let* $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda\in\Lambda}$ *be a TI-DFD of* $\mathcal{K}$*. Then the following hold.*

(a) $\inf_{\lambda\in\Lambda} \kappa_\lambda > 0 \Rightarrow \mathcal{K}^{-1}$ *is bounded.*

(b) $\inf_{\lambda\in\Lambda} \kappa_\lambda = 0$ *and* $\inf_\lambda \|\mathcal{V}_\lambda^*\| > 0 \Rightarrow \mathcal{K}^{-1}$ *unbounded.*

Proof. From the reconstruction formula (2.8) we get

$$\frac{\|(\kappa_\lambda^{-1}\mathcal{V}_\lambda^*)_{\lambda\in\Lambda}(g)\|}{\|(\mathcal{W}^*)^\ddagger\|} \leq \|\mathcal{K}^{-1}g\| \leq \frac{\|\mathcal{W}\|\|\mathcal{V}\|\|g\|}{(\inf_{\lambda\in\Lambda}\kappa_\lambda)}\,. \tag{2.9}$$

The right inequality gives (a). To verify Item (b), suppose $\inf_{\lambda\in\Lambda}\kappa_\lambda = 0$ and $\inf_\lambda\|\mathcal{V}_\lambda^*\| > 0$. Because $\|(\kappa_\lambda^{-1}\mathcal{V}_\lambda^*)_{\lambda\in\Lambda}\| \geq |\kappa_\mu|^{-1}\|\mathcal{V}_\mu^*\|$ for all $\mu \in \Lambda$ this implies that $(\kappa_\lambda^{-1}\mathcal{V}_\lambda^*)_{\lambda\in\Lambda}$ is unbounded. Together with the left inequality in 2.9 this shows the unboundedness of $\mathcal{K}^{-1}$. □

In summary, under the reasonable assumption that $\inf_\lambda\|\mathcal{V}_\lambda^*\| > 0$, the inverse operator $\mathcal{K}^{-1}$ is unbounded if and only if the quasi-singular values κ_λ accumulate at zero.

We note that for a stable inverse problem a TI-DFD has been constructed in [21, Theorem 3.5] in the form of a convolution factorization for the wave equation. An example for a TI-DFD in the ill-posed situation will be constructed in Section 4 for the 1D integration operator.

3 Regularization by filtered TI-DFD

Throughout this section, let $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda\in\Lambda}$ be a TI-DFD for $\mathcal{K}$, and $(w_\lambda)_{\lambda\in\Lambda}$ be a dual frame of $(u_\lambda)_{\lambda\in\Lambda}$. Typically, solving inverse problems of the form (1.1) is unstable (see Theorem 2.11) and hence need to be regularized. For that purpose, we introduce and anlayze the concept of filtered TI-DFD in this section.

3.1 Definition of the filtered TI-DFD

We first recall the definition of regularizing filters. We adopt the Definition from [8] which is slightly more general than the standard definition [9].

Definition 3.1 (Regularizing filter)**.** *A family* $(\Phi_\alpha)_{\alpha>0}$ *of piecewise continuous functions* $\Phi_\alpha\colon (0,\infty) \to \mathbb{R}$ *is called a regularizing filter if the following hold:*

(F1) $\forall\alpha > 0\colon \|\Phi_\alpha\|_\infty < \infty$.

(F2) $\exists C > 0\colon \sup\{|\kappa \Phi_\alpha(\kappa)| \colon \alpha > 0 \wedge \kappa \geq 0\} \leq C$.

(F3) $\forall \kappa \in (0, \infty)\colon \lim_{\alpha \to 0} \Phi_\alpha(\kappa) = 1/\kappa$.

Using the concept of regularizing filters, we study the following filtered versions of the TI-DFD reconstruction formula.

Definition 3.2 (Filtered TI-DFD). *Let $(\Phi_\alpha)_{\alpha>0}$ be a regularizing filter. We call the family $(\mathcal{R}_\alpha^\Phi)_{\alpha>0}$ of operators $\mathcal{R}_\alpha^\Phi \colon \mathbb{Y} \to L^2(\mathbb{R}^d)$ defined by*

$$\mathcal{R}_\alpha^\Phi g \coloneqq \sum_{\lambda \in \Lambda} w_\lambda * (\Phi_\alpha(\kappa_\lambda) \cdot (\mathcal{V}_\lambda^* g)) = \mathcal{W}\big((\Phi_\alpha(\kappa_\lambda) \cdot \mathcal{V}_\lambda^* g)_{\lambda \in \Lambda}\big) \tag{3.1}$$

the filtered TI-DFD according to the filter $(\Phi_\alpha)_{\alpha>0}$ and the TI-DFD $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda \in \Lambda}$.

We first show the well-posedness of filtered TI-DFD.

Proposition 3.3 (Existence and stability). *Let $(\Phi_\alpha)_{\alpha>0}$ be a regularizing filter. For any $\alpha > 0$, the operator $\mathcal{R}_\alpha^\Phi$ as in (3.1) is well defined, linear and bounded with $\|\mathcal{R}_\alpha^\Phi\| \leq \|\Phi_\alpha\|_\infty \|\mathcal{W}\| \|\mathcal{V}\|$.*

Proof. Fix $\alpha > 0$ and let $g \in \operatorname{ran}(\mathcal{K})$. Clearly $\mathcal{R}_\alpha^\Phi$ is a linear operator. From the upper frame property of $(w_\lambda)_{\lambda \in \Lambda}$, the boundedness of the filters Φ_α (see (TI1)) and the boundedness of $\mathcal{V}^*$ we obtain $\|\mathcal{R}_\alpha^\Phi g\| = \|\mathcal{W}\big((\Phi_\alpha(\kappa_\lambda) \cdot \mathcal{V}_\lambda^* g)_{\lambda \in \Lambda}\big)\| \leq \|\Phi_\alpha\|_\infty \|\mathcal{W}\| \|\mathcal{V}\| \|g\|$. This shows, that $\mathcal{R}_\alpha^\Phi$ is well defined and gives the claimed norm estimate. □

3.2 Convergence analysis

For the following, let $(\Phi_\alpha)_{\alpha>0}$ be a regularizing filter and $(\mathcal{R}_\alpha^\Phi)_{\alpha>0}$ be the filtered TI-DFD defined in (3.1). In what follows, we show that the filtered TI-DFD yields a regularization method. To that end we first recall the definition of a regularization method.

Definition 3.4. *Let $(\mathcal{R}_\alpha)_{\alpha>0}$ be a family of bounded linear operators $\mathcal{R}_\alpha \colon \mathbb{Y} \to L^2(\mathbb{R}^d)$, let $g \in \operatorname{ran}(\mathcal{K})$ and $\alpha^* \colon (0, \infty) \times \mathbb{Y} \to (0, \infty)$. The pair $((\mathcal{R}_\alpha)_{\alpha>0}, \alpha^*)$ is a regularization method for the solution of $\mathcal{K} f = g$, if*

$$\lim_{\delta \to 0} \sup\big\{\alpha^*(\delta, g^\delta) \mid g^\delta \in \mathbb{Y} \wedge \|g^\delta - g\| \leq \delta\big\} = 0\,,$$
$$\lim_{\delta \to 0} \sup\big\{\|f - \mathcal{R}_{\alpha^*(\delta, g^\delta)} g^\delta\| \mid g^\delta \in \mathbb{Y} \wedge \|g^\delta - g\| \leq \delta\big\} = 0\,.$$

In this case, function α^* is called admissible parameter choice.

As an auxiliary result we show convergence as $\alpha \to 0$ for exact data.

Proposition 3.5 (Pointwise convergence). *For all $g \in \operatorname{ran}(\mathcal{K})$ we have $\lim_{\alpha \to 0} \mathcal{R}_\alpha^\Phi g = \mathcal{K}^{-1} g$.*

Proof. Let $g \in \operatorname{ran}(\mathcal{K})$ and $f \in \operatorname{dom}(\mathcal{K})$ with $\mathcal{K}f = g$. From the reproducing formula (2.2) we have $f = \sum_{\lambda\in\Lambda} w_\lambda * u_\lambda^* * f$ and from the definition of the filtered TI-DFD together with (TI3) we have $\mathcal{R}_\alpha^\Phi g = \sum_{\lambda\in\Lambda} w_\lambda * (\Phi_\alpha(\kappa_\lambda)\kappa_\lambda \cdot u_\lambda^* * f)$. As a consequence,

$$\|f - \mathcal{R}_\alpha^\Phi g\|^2 \leq \|\mathcal{W}\|^2 \sum_{\lambda\in\Lambda} |1 - \Phi_\alpha(\kappa_\lambda)\kappa_\lambda|^2 \|u_\lambda^* * f\|^2 \leq \|\mathcal{W}\|^2 \|\mathcal{U}^*\|^2 \sup_{\lambda\in\Lambda} |1 - \Phi_\alpha(\kappa_\lambda)\kappa_\lambda|^2 \|f\|^2 . \quad (3.2)$$

According to (F3), $\lim_{\alpha\to 0}|1-\Phi_\alpha(\kappa_\lambda)\kappa_\lambda| = 0$ pointwise and according to (F2), $\sup_{\lambda\in\Lambda}|1-\Phi_\alpha(\kappa_\lambda)\kappa_\lambda|$ is bounded independently of α. Application of the dominated convergence theorem to the sum in (3.2) yield $\|f - \mathcal{R}_\alpha^\Phi g\| \to 0$. □

As a consequence of Propositions 3.3 and 3.5 we derive the following main regularization result. For convenience of the reader we restate all assumptions on the operator and the filtered DFD in that theorem.

Theorem 3.6 (Filtered TI-DFD is regularization method)**.** *Let $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda\in\Lambda}$ be a TI-DFD for the closed linear operator $\mathcal{K}\colon \operatorname{dom}(\mathcal{K}) \subseteq L^2(\mathbb{R}^d) \to \mathbb{Y}$, $(w_\lambda)_{\lambda\in\Lambda}$ a dual frame of $(u_\lambda)_{\lambda\in\Lambda}$, $(\Phi_\alpha)_{\alpha>0}$ a regularizing filter and $(\mathcal{R}_\alpha^\Phi)_{\alpha>0}$ defined by (3.1). Let $\alpha^*\colon (0,\infty) \to (0,\infty)$ satisfy*

$$\lim_{\delta\to 0} \alpha^*(\delta) = \lim_{\delta\to 0} \delta \|\Phi_{\alpha^*(\delta)}\|_\infty = 0 . \quad (3.3)$$

Then $((\mathcal{R}_\alpha^\Phi)_{\alpha>0}, \alpha^*)$ is a regularization method for $\mathcal{K}f = g$ for any $g \in \operatorname{ran}(\mathcal{K})$.

Proof. According to Propositions 3.3 and 3.5, $(\Phi_\alpha)_{\alpha>0}$ is a family of bounded linear operators that converges pointwise to $\mathcal{K}^{-1}$ on $\operatorname{dom}(\mathcal{K}^{-1})$. Using [9, Propositions 3.4, 3.7] the pair $((\mathcal{R}_\alpha^\Phi)_{\alpha>0}, \alpha^*)$ is a regularization method if $\alpha^*(\delta), \delta\|\mathcal{R}_{\alpha^*(\delta)}\| \to 0$ as $\delta \to 0$. The estimate $\|\mathcal{R}_\alpha^\Phi\| \leq \|\Phi_\alpha\|_\infty \|\mathcal{W}\| \|\mathcal{V}\|$ derived in Proposition 3.3 then yields the claim. □

3.3 Convergence rates

As the next result we derive convergence rates for filtered TI-DFD, which gives quantitative estimates for the reconstruction error $\|f_\star - \mathcal{R}_\alpha g^\delta\|$. Due to the ill-posedness of (1.1) such estimates require additional assumptions on the exact unknown for any reconstruction method. In the following we write $\mathbf{h} = (h_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda, L^2(\mathbb{R}^d))$. We derive convergence rates under the following assumptions on the exact unkown $f_\star \in L^2(\mathbb{R}^d)$ and the regularizing filter $(\Phi_\alpha)_{\alpha>0}$.

Assumption 3.7 (Convergence rates conditions)**.** *For $\mu, \rho > 0$ suppose:*

(R1) $\exists \mathbf{h} \in \ell^2(\Lambda, L^2(\mathbb{R}^d))\colon \|\mathbf{h}\| \leq \rho \wedge \forall \lambda \in \Lambda\colon u_\lambda^* * f_\star = \kappa_\lambda^{2\mu} h_\lambda$.

(R2) $\|\Phi_\alpha\|_\infty = \mathcal{O}(\alpha^{-1/2})$ *as* $\alpha \to 0$.

(R3) $\forall \alpha > 0\colon \sup\{\kappa^{2\mu}|1 - \kappa\Phi_\alpha(\kappa)| \mid \kappa \in (0,\infty)\} \leq C_\mu \alpha^\mu$.

Source condition (R1) is an abstract smoothness condition relating the element to be reconstructed with the ill-posedness of the operator characterized by the quasi-singular values. Conditions (R2)-(R3) restrict the class of filters yielding a desired rate. For example, it is well known that the hard truncation filter satisfies these conditions for all $\mu > 0$.

We have the following result.

Theorem 3.8 (Convergence rates)**.** *Suppose $f_\star$ and $(\Phi_\alpha)_{\alpha>0}$ satisfy (R1)-(R3) and make the parameter choice $\alpha = \alpha^*(\delta, g^\delta) \asymp (\delta/\rho)^{2/(2\mu+1)}$. Then there exists a constant c_μ independent of $f_\star, \rho$ such that for all $g^\delta \in \mathbb{Y}$ with $\|g^\delta - \mathcal{K} f_\star\| \leq \delta$,*

$$\|f_\star - \mathcal{R}^\Phi_{\alpha^*}(g^\delta)\| \leq c_\mu \delta^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} . \tag{3.4}$$

Proof. From (R1) and the estimate $\|g^\delta - \mathcal{K} f_\star\| \leq \delta$ we obtain

$$\begin{aligned}
\|\mathcal{R}^\Phi_\alpha g^\delta - f_\star\| &\leq \|\mathcal{R}^\Phi_\alpha (g^\delta - \mathcal{K} f_\star)\| + \|\mathcal{R}^\Phi_\alpha \mathcal{K} f_\star - f_\star\| \\
&\leq \|\mathcal{R}^\Phi_\alpha\| \delta + \|\sum_{\lambda\in\Lambda} w_\lambda * ((1 - \Phi_\alpha(\kappa_\lambda)\kappa_\lambda) \cdot u^*_\lambda * f_\star)\| \\
&\leq \|\Phi_\alpha\|_\infty \|\mathcal{W}\| \|\mathcal{V}\| \delta + \|\mathcal{W}\| \big(\sum_{\lambda\in\Lambda} |1 - \Phi_\alpha(\kappa_\lambda)\kappa_\lambda|^2 \|u_\lambda * f_\star\|^2\big)^{1/2} \\
&\leq \|\Phi_\alpha\|_\infty \|\mathcal{W}\| \|\mathcal{V}\| \delta + \|\mathcal{W}\| \, \big(\sum_{\lambda\in\Lambda} |(1 - \Phi_\alpha(\kappa_\lambda)\kappa_\lambda) \kappa_\lambda^{2\mu}|^2 \, \|h_\lambda\|^2\big)^{1/2} \\
&\leq c_1 \alpha^{-1/2} \delta + C_\mu \|\mathcal{W}\| \alpha^\mu \rho \, .
\end{aligned}$$

With the parameter choice $\alpha = \alpha^* \asymp (\delta/\rho)^{2/(2\mu+1)}$ this yields (3.4). □

3.4 Order optimality

We next proof that the convergence rates obtained in Theorem 3.8 are optimal. For that purpose, for $\mathcal{M} \subseteq \operatorname{dom}(\mathcal{K})$ and $\mathcal{R} \colon \mathbb{Y} \to L^2(\mathbb{R}^d)$ define

$$E(\mathcal{M}, \delta, \mathcal{R}) \coloneqq \sup\{\|\mathcal{R}(g^\delta) - f\| \mid f \in \mathcal{M} \wedge g^\delta \in \mathbb{Y} \wedge \|\mathcal{K} f - g^\delta\| \leq \delta\} \tag{3.5}$$

$$\epsilon(\mathcal{M}, \delta) \coloneqq \sup\{\|f\| \mid f \in \mathcal{M} \wedge \|\mathcal{K} f\| \leq \delta\} \, . \tag{3.6}$$

The quantity $E(\mathcal{M}, \delta, \mathcal{R})$ is the worst case reconstruction error using the reconstruction method $\mathcal{R}$ under the a-priori assumption $f \in \mathcal{M}$ and the error bound $\|\mathcal{R} g^\delta - f\| \leq \delta$. If $\mathcal{R}(0) = 0$ one readily verifies that $E(\mathcal{M}, \delta, \mathcal{R}) \geq \epsilon(\mathcal{M}, \delta)$. A family $(\mathcal{R}^\delta)_{\delta>0}$ is called order optimal on $\mathcal{M}$, if $E(\mathcal{M}, \delta, \mathcal{R}^\delta) \leq c \cdot \epsilon(\mathcal{M}, \delta)$ for some $c > 0$ and sufficiently small δ.

Theorem 3.8 gives an upper bound for the worst case error of filtered TI-DFD on

$$\mathcal{M}_{\mu,\rho} \coloneqq \{f \in \operatorname{dom}(\mathcal{K}) \mid \exists \mathbf{h} \colon \|\mathbf{h}\| \leq \rho \wedge \forall \lambda \colon u^*_\lambda * f = \kappa_\lambda^{2\mu} h_\lambda\} \, . \tag{3.7}$$

In order to show that filtered TI-DFD is order optimal on $\mathcal{M}_{\mu,\rho}$, we will bound $\epsilon(\mathcal{M}_{\mu,\rho}, \delta)$ from below. Such an estimate is derived in the following Theorem 3.9, at least for a sequence of noise levels tending to zero. Note that $\epsilon(\mathcal{M}_{\mu,\rho}, \delta)$ is monotonically increasing in δ and therefore we have an error bound for any sufficiently small δ.

Theorem 3.9 (Order optimality of filtered TI-DFD)**.** *Let $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda\in\Lambda}$ be a TI-DFD for $\mathcal{K}$ such that 0 is an accumulation point of $(\kappa_\lambda)_{\lambda>0}$ and assume there exists $(e_\lambda)_{\lambda\in\Lambda} \in L^2(\mathbb{R}^d)^\Lambda$ with $u_\lambda^* * e_{\lambda'} = 0$ for $\lambda \neq \lambda'$ and $\|u_\lambda^* * e_\lambda\| = 1$. Then, for some sequence $(\delta_n)_{\in\mathbb{N}}$ with $\delta_n \to 0$,*

$$\epsilon(\mathcal{M}_{\mu,\rho}, \delta_n) \geq \|\mathcal{U}^*\|^{-1} \|(\mathcal{V}^*)^\ddagger\|^{-2\mu/(2\mu+1)} \cdot \delta_n^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} . \tag{3.8}$$

Proof. After extracting a subsequence we can assume $\Lambda = \mathbb{N}$ and that $(\kappa_\lambda)_{\lambda\in\mathbb{N}}$ converges to zero. For any $n \in \mathbb{N}$ define $f^{(n)} := \rho\kappa_n^{2\mu} e_n$. Then $u_\lambda^* * f^{(n)} = \kappa_\lambda^{2\mu} h_\lambda^{(n)}$ where $h_\lambda^{(n)} = 0$ for $\lambda \neq n$ and $h_n^{(n)} = \rho \cdot (u_n^* * e_n)$ with $\|\mathbf{h}^{(n)}\| = \|h_\lambda^{(n)}\| = \rho$. In particular, $f^{(n)} \in \mathcal{M}_{\mu,\rho}$. Moreover,

$$\|\mathcal{K}f^{(n)}\| \leq \|(\mathcal{V}^*)^\ddagger\| \bigl(\sum_{\lambda\in\Lambda} \|\mathcal{V}_\lambda^* \mathcal{K} f^{(n)}\|^2\bigr)^{1/2} = \|(\mathcal{V}^*)^\ddagger\| \bigl(\sum_{\lambda\in\Lambda} \kappa_\lambda^2 \|u_\lambda^* * f^{(n)}\|^2\bigr)^{1/2} = \|(\mathcal{V}^*)^\ddagger\| \kappa_n^{2\mu+1} \rho ,$$

$$\|f^{(n)}\| \geq \|\mathcal{U}^*\|^{-1} \bigl(\sum_{\lambda\in\Lambda} \|u_\lambda^* * f^{(n)}\|^2\bigr)^{1/2} = \|\mathcal{U}^*\|^{-1} \kappa_n^{2\mu} \rho .$$

With $\delta_n := \|(\mathcal{V}^*)^\ddagger\| \kappa_n^{2\mu+1} \rho$, the above estimates imply $\|\mathcal{K}f^{(n)}\| \leq \delta_n$ and

$$\|f^{(n)}\| \geq \|\mathcal{U}^*\|^{-1} (\delta_n \|(\mathcal{V}^*)^\ddagger\|^{-1} \rho^{-1})^{2\mu/(2\mu+1)} \rho = \|\mathcal{U}^*\|^{-1} \|(\mathcal{V}^*)^\ddagger\|^{-2\mu/(2\mu+1)} \delta_n^{2\mu/(2\mu+1)} \rho^{1/(2\mu+1)} .$$

Because we have $f^{(n)} \in \mathcal{M}_{\mu,\rho}$ and $\epsilon(U_{\nu,\rho}, \delta_n) \geq \|f^{(n)}\|$ which yields (3.8). □

3.5 Examples for filtered DFDs

We conclude this section by giving two representative examples for regularizing filters and corresponding filtered TI-DFD, namely truncated TI-DFD and Tikhonov-filtered TI-DFD. In particular we show that the convergence rates conditions are satisfied for all $\mu > 0$ in case of truncated TI-DFD and for $\mu \leq 1$ in case of Tikhonov-filtered TI-DFD. Concrete examples of TI-DFDs for the 1D integration operator are discussed in the following section.

Example 3.10 (Truncated TI-DFD)**.** *For $\alpha > 0$ consider the truncation filter $\Phi_\alpha^{(1)} := \chi_{[\alpha^{1/2},\infty)}$. Clearly, Conditions (F1)-(F3) are satisfied with $C = 1$ and $(\Phi_\alpha^{(1)})_{\alpha>0}$ is a regularizing filter. Furthermore, $\sup\{\kappa^{2\mu}|1 - \kappa\Phi_\alpha^{(1)}(\kappa)| \mid \kappa > 0\} = \sup\{\kappa^{2\mu}|1 - \kappa\Phi_\alpha^{(1)}(\kappa)| \mid \kappa^2 < \alpha\} = \alpha^\mu$ for $\alpha, \mu > 0$. Hence (R2), (R3) are satisfied. The corresponding truncated TI-DFD becomes*

$$\mathcal{R}_\alpha^{(1)}(y) := \sum_{\kappa_\lambda^2 \geq \alpha} w_\lambda^* * (\kappa_\lambda^{-1} \cdot (\mathcal{V}_\lambda^* g)) . \tag{3.9}$$

The considerations above allow application of Theorem 3.6 yielding convergence, and Theorem 3.8 *providing the convergence rate* $\|f_\star - \mathcal{R}_\alpha^{(1)} g^\delta\| = \mathcal{O}(\delta^{2\mu/(2\mu+1)})$ *under (R1).*

Next we consider the Tikhonov filter that already appeared in the introduction in the context of classical Tikhonov regularization expressed in terms of the SVD.

Example 3.11 (Tikhonov-filtered TI-DFD)**.** *For* $\alpha > 0$ *consider the Tikhonov filter* $\Phi_\alpha^{(2)}(\kappa) := \kappa/(\kappa^2+\alpha)$*. Then* $|\kappa \Phi_\alpha^{(2)}(\kappa)| = |\kappa^2/(\kappa^2+\alpha)| \leq 1$ *and* $\lim_{\alpha\to 0} \Phi_\alpha^{(2)}(\kappa) = 1/\kappa$*. Further,* $\Phi_\alpha^{(2)}$ *is bounded, takes its maximum at* $\kappa^2 = \alpha$ *and* $\|\Phi_\alpha^{(2)}\|_\infty = \alpha^{-1/2}/2$*. Hence conditions(F1)-(F3) are satisfied and* $(\Phi_\alpha^{(2)})_{\alpha>0}$ *is a regularizing filter. Moreover, one shows that the convergence rates conditions for Theorem 3.8 are satisfied for* $\mu \in (0,1]$*. However (R3) is not satisfied for* $\mu > 1$*, which means that the Tikhonov filter has qualification* $\mu = 1$ *(see the discussion on [9, p. 76]). Above considerations show that Tikhonov-filtered TI-DFD*

$$\mathcal{R}_\alpha^{(2)}(y) := \sum_{\lambda\in\Lambda} w_\lambda^* * \Big(\frac{\kappa_\lambda}{\kappa_\lambda^2+\alpha} \cdot (\mathcal{V}_\lambda^* g)\Big) \tag{3.10}$$

is a regularization method. Moreover, for elements $f_\star$ *satisfying the source condition (R1) the convergence rate* $\|f_\star - \mathcal{R}_\alpha^{(2)} g^\delta\| = \mathcal{O}(\delta^{2\mu/(2\mu+1)})$ *holds.*

Further examples of filtered TI-DFDs can be constructed via any known regularizing filter used in standard SVD-based regularization methods [9].

4 Application: stable differentiation

In this section we apply the concept of filtered TI-DFD to stable differentiation, the inverse problem associated to 1D integration. In particular, we use TI-wavelets as underlying TI-frame. While related approaches are known for denoising, we are not aware of such methods in the context of general inverse problems. We consider the one-dimensional integration operator as an unbounded operator on $L^2(\mathbb{R})$. While the integration operator would be bounded on a bounded domain, working on $\mathbb{R}$ allows to use the concept of TI wavelets and moreover preserves the translation invariance of integration.

4.1 Integration operator on $L^2(\mathbb{R})$

Let $C_\diamond(\mathbb{R})$ denote the space of all continuous functions with compact support and zero integral $\int_\mathbb{R} f = 0$. For functions $f \in C_\diamond(\mathbb{R})$ define the primitive $\mathcal{I}_\diamond f \colon \mathbb{R} \to \mathbb{C}$ by

$$\forall x \in \mathbb{R}\colon \quad (\mathcal{I}_\diamond f)(x) := \int_{-\infty}^{x} f(t)\,\mathrm{d}t\,. \tag{4.1}$$

Note that for all $f \in C_c(\mathbb{R})$, the space of all continuous functions with compact support, the primitive $x \mapsto \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ becomes constant for sufficiently large x. Therefore, the additional assumption of zero integral is necessary and sufficient for the primitive being square integrable.

Lemma 4.1 (Integration operator on $C_\diamond(\mathbb{R})$)**.** *Operator* $\mathcal{I}_\diamond \colon C_\diamond(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}) \colon f \mapsto \mathcal{I}_\diamond f$ *is well defined, linear, densely defined and unbounded. Moreover, for all* $f \in C_\diamond(\mathbb{R})$*, functions* $\mathcal{F}f$ *and* $\mathcal{F}\mathcal{I}_\diamond f$ *are continuous with* $(\mathcal{F}\mathcal{I}_\diamond f)(\omega) = (i\omega)^{-1}(\mathcal{F}f)(\omega)$*.*

Proof. As noted above, $\mathcal{I}_\diamond f \in L^2(\mathbb{R})$ for any $f \in C_\diamond(\mathbb{R})$ and therefore $\mathcal{I}_\diamond$ is well defined and clearly linear. In order to show that $\mathcal{I}_\diamond$ is densely defined it is sufficient to show that $C_\diamond(\mathbb{R})$ is dense with respect to the L^2-norm in $C_c(\mathbb{R})$. For that purpose, let $g \in C_c(\mathbb{R})$ and assume without loss of generality that $\operatorname{supp}(g) \subseteq [-1, 0]$. For any $n \in \mathbb{N}$ define $g_n(x) = g(x)$ for $x < 0$, $g_n(x) = g(-x/n)/n$ for $x \in [0, n]$ and $g_n(x) = 0$ otherwise. Then $g_n \in C_\diamond(\mathbb{R})$ and $\|g - g_n\|^2 \leq \|g\|^2/n \to 0$. Hence $C_\diamond(\mathbb{R})$ is dense in $C_c(\mathbb{R})$. Furthermore $f, \mathcal{I}_\diamond f$ are integrable and therefore $\mathcal{F}f, \mathcal{F}\mathcal{I}_\diamond$ are continuous functions defined by the standard Fourier integral. Integration by parts shows $\mathcal{F}\mathcal{I}_\diamond f(\omega) = \int_\mathbb{R} e^{-ix\omega} \mathcal{I}_\diamond f(x)\,\mathrm{d}x = (i\omega)^{-1} \int_\mathbb{R} e^{-ix\omega} f(x)\,\mathrm{d}x = (i\omega)^{-1}\mathcal{F}f(\omega)$. With the unboundedness of ω this shows the unboundedness of $\mathcal{I}_\diamond$. □

Below we extend $\mathcal{I}_\diamond$ to a closed operator on $L^2(\mathbb{R})$ with dense but non-closed domain. For that purpose we recall some basic facts on the adjoint and closure of unbounded operators that we will use for our purpose.

Remark 4.2 (Adjoint and closure of unbounded operators)**.** *For a densely defined potentially unbounded operator* $\mathcal{K} \colon \operatorname{dom}(\mathcal{K}) \subseteq \mathbb{X} \to \mathbb{Y}$ *one defines the adjoint domain* $\operatorname{dom}(\mathcal{K}^*)$ *as the set of all* $g \in \mathbb{Y}$ *such that* $\langle \mathcal{K}(\cdot), g\rangle \colon f \mapsto \langle \mathcal{K}f, g\rangle$ *is bounded. According to the Riesz representation theorem for any* $g \in \operatorname{dom}(\mathcal{K}^*)$ *there exists a unique element* $\mathcal{K}^* g \in \mathbb{X}$ *such* $\langle \mathcal{K}(\cdot), g\rangle = \langle \cdot, \mathcal{K}^* g\rangle$*, which defines the adjoint* $\mathcal{K}^* \colon \operatorname{dom}(\mathcal{K}^*) \subseteq \mathbb{Y} \to \mathbb{X}$*. A linear operator is called closable if it has an extension to a closed linear operator. It is known that* $\mathcal{K}$ *is closable if and only if* $\operatorname{dom}(\mathcal{K}^*)$ *is dense, in which case* $\mathcal{K}^{**}$ *is the closure of* $\mathcal{K}$*.*

Following Remark 4.2, we next extend $\mathcal{I}_\diamond$ to a closed operator $\mathcal{I} \colon \operatorname{dom}(\mathcal{I}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and further summarize some basic properties that will be of later use. We call this extension the integration operator on $L^2(\mathbb{R})$.

Proposition 4.3 (Integration operator on $L^2(\mathbb{R})$)**.**

(a) $\mathcal{I}_\diamond$ *has a closed extension* $\mathcal{I} \colon \operatorname{dom}(\mathcal{I}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$*.*

(b) $\operatorname{dom}(\mathcal{I}) = \operatorname{dom}(\mathcal{I}^*) = \{f \in L^2(\mathbb{R}) \mid (i\omega)^{-1}\mathcal{F}f(\omega) \in L^2(\mathbb{R})\}$*.*

(c) $\forall f \in \operatorname{dom}(\mathcal{I}) \colon \mathcal{F}\mathcal{I}f(\omega) = (i\omega)^{-1}\mathcal{F}f(\omega)$*.*

(d) $\mathcal{I}^* = -\mathcal{I}$*.*

(e) $\mathcal{I}$ *is injective with dense range* $\operatorname{ran}(\mathcal{I}) = \{g \in L^2(\mathbb{R}) \mid (i\omega)\mathcal{F}g(\omega) \in L^2(\mathbb{R})\}$*.*

(f) $\forall g \in \operatorname{ran}(\mathcal{I}) \colon \mathcal{F}\mathcal{I}^{-1}g(\omega) = (i\omega)\mathcal{F}g(\omega)$*.*

Proof. By Plancherel's theorem and Lemma 4.1 we have $\langle \mathcal{I}_\diamond f, g \rangle = (2\pi)^{-1} \int_{\mathbb{R}} (\mathcal{F} f)(i\omega)^{-1} \overline{\mathcal{F} g}$. Hence $\langle \mathcal{I}_\diamond(\cdot), g \rangle$ is bounded iff $(i\omega)^{-1} \mathcal{F} g \in L^2(\mathbb{R}^2)$ and $\operatorname{dom}(\mathcal{I}^*) = \{g \in L^2(\mathbb{R}) \mid (i\omega)^{-1} \mathcal{F} g(\omega) \in L^2(\mathbb{R})\}$. In particular $\operatorname{dom}(\mathcal{I}^*)$ is dense which gives (a). Similar arguments show $\operatorname{dom}(\mathcal{I}) = \operatorname{dom}(\mathcal{I}^*)$ and (b)-(d). Items (e), (f) are immediate consequences of (b), (c). □

Our aim is the stable inversion of the integration operator $\mathcal{I}$, which is equivalent to the stable evaluation of differentiation. For that purpose we use the concept of filtered TI-DFDs introduced in this paper. In particular we use the TI-wavelet transform.

4.2 TI-DFDs for the integration operator

If $(u_\lambda, \mathcal{V}_\lambda^*, \kappa_\lambda)_{\lambda \in \Lambda}$ is a TI-DFD for $\mathcal{I}$, then the translation invariance of $\mathcal{I}$ implies the translation invariance of $\mathcal{V}_\lambda^*$. We will therefore consider in the following the case where $\mathcal{V}_\lambda^* g = v_\lambda^* * g$ for a TI-frame $(v_\lambda)_{\lambda \in \Lambda}$ of $L^2(\mathbb{R})$. We will also refer to $(u_\lambda, v_\lambda, \kappa_\lambda)_{\lambda \in \Lambda}$ as TI-DFD.

Before constructing a wavelet based TI-DFD we start by necessary conditions to be satisfied by the TI-DFDs.

Proposition 4.4 (Necessary condition for TI-DFD)**.** *Let $(u_\lambda, v_\lambda, \kappa_\lambda)_{\lambda \in \Lambda}$ is TI-DFD for $\mathcal{I}$. Then u_λ are weakly differentiable with weak derivative $\partial_x u_\lambda = -v_\lambda / \kappa_\lambda$ and*

$$\forall \lambda \in \Lambda \colon \quad \hat{v}_\lambda(\omega) = \kappa_\lambda \cdot (-i\omega) \cdot \hat{u}_\lambda(\omega) \,. \tag{4.2}$$

Proof. According to (TI3) we have $v_\lambda^* * (\mathcal{I} f) = \kappa_\lambda \cdot (v_\lambda^* * f)$ for all $f \in \operatorname{dom}(\mathcal{I})$. With the Fourier convolution theorem and the Fourier representation of $\mathcal{I}$ we get $(i\omega)^{-1} \cdot (\overline{\mathcal{F} v_\lambda}) \cdot \mathcal{F} f = \kappa_\lambda \cdot (\overline{\mathcal{F} u_\lambda}) \cdot \mathcal{F} f$. Since $\operatorname{dom}(\mathcal{I})$ is dense this gives (4.2). According to (4.2), $(i\omega) \cdot \hat{u}_\lambda$ are in $L^2(\mathbb{R}^2)$ and u_λ has weak derivative $\partial_x u_\lambda = -v_\lambda / \kappa_\lambda$. □

Proposition (4.4) states that a necessary condition for $(u_\lambda)_{\lambda \in \Lambda}$ to be part of TI-DFD is that all u_λ are weakly differentiable and that there exist constants $\kappa_\lambda > 0$ such that $(-\kappa_\lambda \partial_x u_\lambda)_{\lambda \in \Lambda}$ again is a TI-frame. According to Proposition 2.3 this means that there are constants $A_\diamond, B_\diamond$ such that $2\pi A_\diamond \leq \sum_{\lambda \in \Lambda} |\kappa_\lambda \omega \hat{u}_\lambda(\omega)|^2 \leq 2\pi B_\diamond$. Clearly there is a wide range of TI-frames satisfying these properties. In the following theorem we provide a class of examples using TI-wavelets $(u_j)_{j \in \mathbb{Z}}$, given of the form

$$\forall j \in \mathbb{Z} \colon \quad u_j(x) = 2^j u(2^j x), \tag{4.3}$$

$$\forall j \in \mathbb{Z} \colon \quad v_j(x) = -\partial_x u(2^j x) \,. \tag{4.4}$$

In this case we call the elements v_j TI-vaguelettes and the corresponding TI-DFD $(u_j, v_j, 2^{-j})_{j \in \mathbb{Z}}$ a translation invariant wavelet-vaguelette decomposition (TI-WVD).

Theorem 4.5 (TI-WVD for $\mathcal{I}$)**.** *Let $(u_j)_{j\in\mathbb{Z}}$ be a TI-wavelet frame of the form* (4.3) *such that* $\operatorname{supp}(u) \subseteq \{\omega \in \mathbb{R} \mid a \leq |\omega| \leq b\}$ *for some $a, b > 0$. Then with $(v_j)_{j\in\mathbb{Z}}$ as in* (4.4)*, the family $(u_j, v_j, 2^{-j})_{j\in\mathbb{Z}}$ is a TI-DFD for $\mathcal{I}$, named TI-WVD for the integration operator.*

Proof. According to Proposition (4.4) it remains to show that $(2^{-j}v_j)_{j\in\mathbb{Z}}$ is TI-frame. Equivalently, we have to show $2\pi \asymp \sum_{\lambda\in\Lambda} |(2^{-j}\omega) \cdot \hat{u}_j(\omega)|^2$. Let $A, B > 0$ be the frame bounds of the $(u_j)_{j\in\mathbb{Z}}$. Since $\operatorname{supp}(\hat{u}) \subseteq \{\omega \in \mathbb{R} \mid a \leq |\omega| \leq b\}$, the scaled versions $\hat{u}_j(\omega)$ have support in $\{\omega \in \mathbb{R} \mid a2^j \leq |\omega| \leq b2^j\}$. Therefore, we have $2^j a|\hat{u}_j(\omega)| \leq |\omega \hat{u}_j(\omega)| \leq 2^j b|\hat{u}_j(\omega)|$. Together with the frame property of $(u_j)_{j\in\mathbb{Z}}$ this gives $2\pi a^2 A \leq \sum_{\lambda\in\Lambda} 2^{-2j} |\omega \mathcal{F} u_j(\omega)|^2 \leq 2\pi b^2 B$ and concludes the proof. □

Note the we made the restriction in Theorem 4.5 to compactly supported wavelets in order to avoid technical difficulties and to focus on the main ideas. Similar results can clearly be derived under weaker assumptions guaranteeing that $2\pi \asymp \sum_{\lambda\in\Lambda} |(2^{-j}\omega) \cdot \hat{u}(2^{-j}\omega)|^2$.

4.3 Numerical realization

For the numerical simulations presented below we use the Tikhonov-type filter (see Example 3.11). The corresponding Tikhonov filtered TI-WVD reads

$$\mathcal{R}_\alpha^{(2)}(g) \coloneqq \sum_{j\in\mathbb{Z}} w_\lambda^* * \left(\frac{2^{-j}}{2^{-2j} + \alpha} \cdot (v_j * g) \right) \tag{4.5}$$

The instability of differentiation is reflected in the decreasing quasi-singular values $\kappa_j = 2^{-j}$ with increasing j. The filtered TI-WVD stabilizes the inversion by replacing the inverse quasi-singular coefficients $1/2^{-j}$ by the Tikhonov filtered approximations $2^{-j}/(2^{-2j} + \alpha)$.

The TI-vaguelette coefficients can be computed by $v_j^* * g = -2^{-j}(\partial_x u_j * g) = -2^{-j}\partial_x(u_j * g)$. The latter form has the advantage that any existing implementation for computing the TI-wavelet coefficients $u_j * g$ can be used for its evaluation. Such a strategy will be employed in this paper. In summary, the Tikhonov filtered TI-WVD reconstruction (4.5) can be implemented by following Algorithm.

Algorithm 4.6 (Tikhonov-filtered TI-WVD reconstruction)**.**
Input: Noisy data g^δ.
Parameters: Mother wavelets u, w and regularization parameter $\alpha > 0$.
Output: Regularized rconstruction f_α^δ.

(A1) Compute the auxiliary TI-wavelet coefficients $(\mathcal{U}_j^* g^\delta)_{j\in\mathbb{Z}} = (u_j^* * g^\delta)_{j\in\mathbb{Z}}$.

(A2) Obtain the TI-vaguelette coefficients by $(d_j)_{j\in\mathbb{Z}} = (-2^{-j}\partial_x \mathcal{U}_j^* g^\delta)_{j\in\mathbb{Z}}$.

(A3) Multiplication with Tikhonov-filter coefficients: $(c_j)_{j\in\mathbb{Z}} \coloneqq (2^{-j}/(2^{-2j} + \alpha^2) \cdot d_j)_{j\in\mathbb{Z}}$.

(A4) Application of TI-wavelet synthesis $f_\alpha^\delta \coloneqq \mathcal{W}((c_j)_{j\in\mathbb{Z}})$.

For the numerical simulation, functions f, g and corresponding TI-coefficients are given on the interval $[0,1]$ and discretized using $N = 512$ equidistant sample points. The TI-wavelet operators $\mathcal{U}^*$, $\mathcal{W}$ are numerically computed with the PyWavelet package version 1.1.1 [14] in Python 3.8.8. The standard decimated Tikhonov-filtered WVD takes a form similar to (4.5), where the inner and outer convolution are replaced by the downsampled and upsampled convolution, respectively (see Remark 2.7) and Algorithm 4.6 is adjusted accordingly.

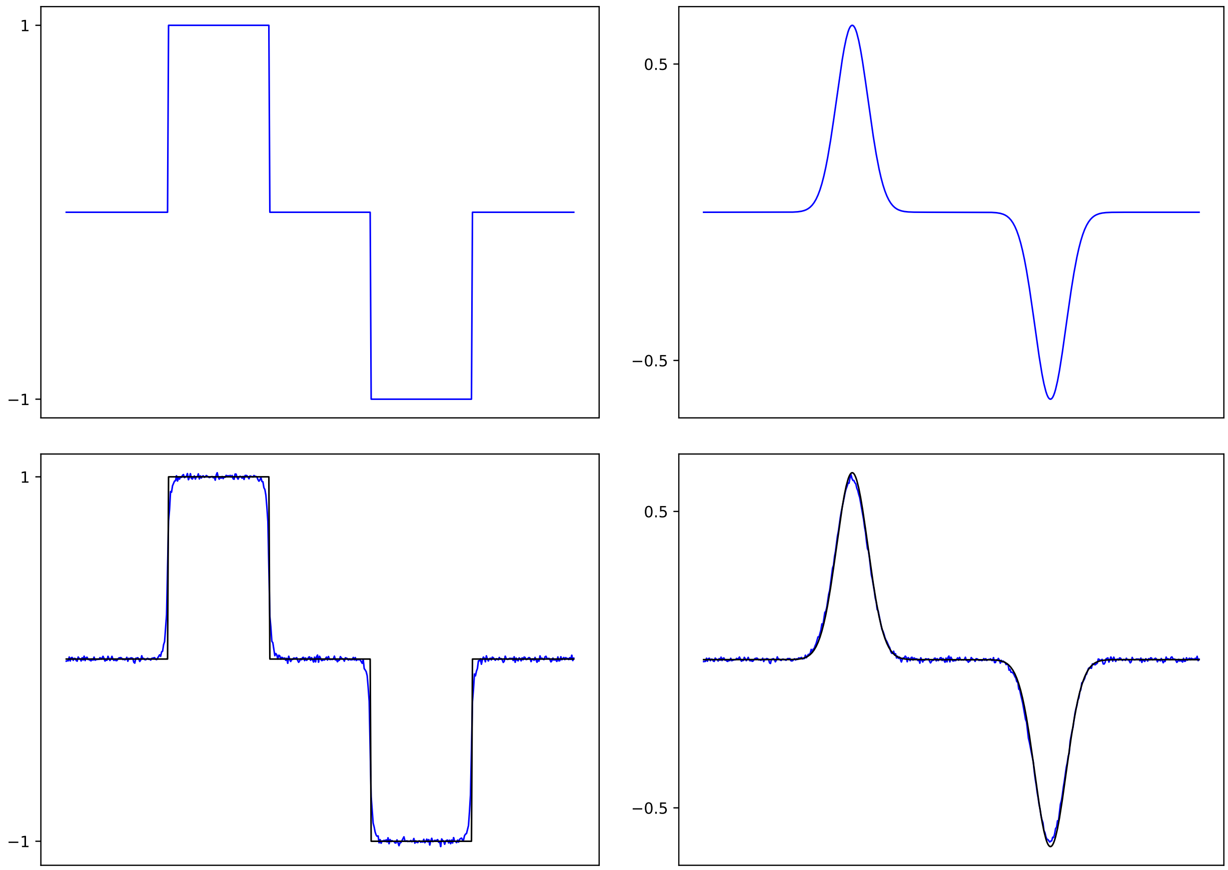

Figure 1: *Reconstructions via TI-WVD.* Top: Original signals. Bottom: Reconstructions from noisy data using filtered TI-WVD.

4.4 Numerical examples

In this subsection we present reconstruction results for the (Tikhonov) filtered TI-WVD and compare with the standard decimated WVD. We first demonstrate the feasibility on the two signals shown in the top row in Figure 1. We add Gaussian noise to the data and take $g^\delta = \mathcal{I}f + \eta$ where η is Gaussian white noise with standard deviation $\sigma = 0.03$. Filtered TI-WVD (4.5) is applied afterwards with regularization parameter $\alpha = 0.01$ for the piecewise constant and $\alpha = 0.025$ for the smooth phantom. Reconstruction results are shown in the bottom row in Figure 1. The relative ℓ^2-reconstruction error $\|f_\star - f_\alpha^\delta\|_2/\|f_\star\|_2$ is 0.0095 for the piecewise constant and 0.0016 for the smooth phantom.

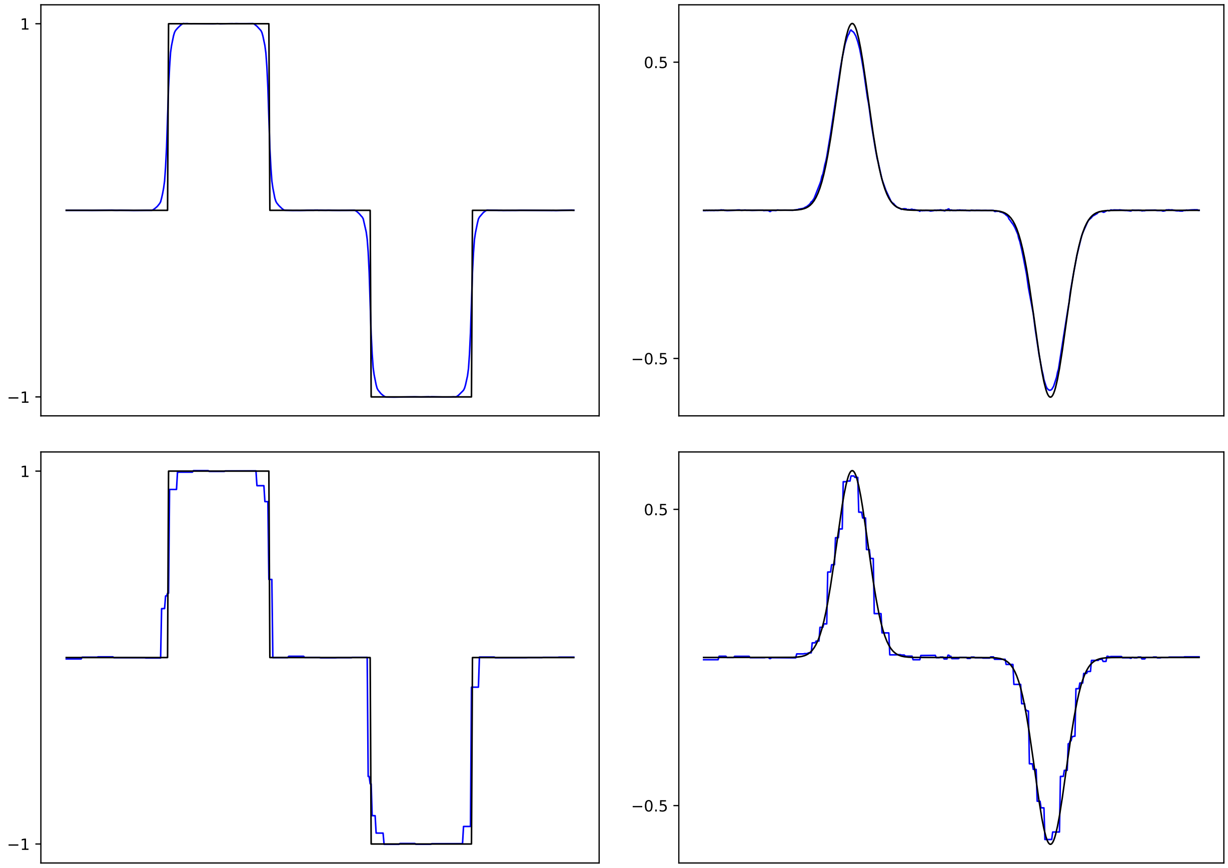

Figure 2: *Comparison of TI-WVD and WVD.* Top: Reconstruction using proposed TI-WVD. Bottom Reconstruction with decimated WVD.

As a second example we compare the TI-WVD with the standard decimated WVD. For that purpose we include coefficient thresholding which is known to optimally remove Gaussian white noise [7, 1, 12]. Since we are dealing with multi-scale decompositions we choose level dependent thresholds $t = 2^{-j}\beta$, for some fixed $\beta > 0$ and replace the vaguelette coefficients by $\mathrm{soft}(2^{-j}\beta, v_j^* * g)$ with soft-thresholding function $\mathrm{soft}(t, x) := \mathrm{sign}(x)\max\{0, |x| - t\}$. Figure 2 shows reconstruction from noisy measurements using thresholded TI-WVD and thresholded WVD. The thresholding parameter was chosen as $\beta = 0.05$ and $\beta = 0.005$, respectively. The results clearly suggest, that the TI-WVD yields more reliable reconstruction, without typical wavelet artifacts of decimated WVD. This is inline with reported results for the simple denoising task.

5 Conclusion

This work introduced the translation invariant frame decomposition (TI-DFD) as novel tool for solving ill-posed inverse problems. We showed that filtered TI-DFDs yields a regularization method with order optimal rates. Advantage of the filtered TI-DFD over iterative regularization methods is the explicit form and thus an efficient implementation. Opposed to the non-TI counterparts the TI-DFD does not suffer from typical artifacts known for example for wavelet thresholding. As an illustrative example we constructed a TI-DFD for 1D integration using wavelet frames (TI-WVD). We demonstrated that filtered TI-WVD works well and improves over the standard WVD with significantly reduced wavelet artifacts. Potential aspects of future research are the analysis of nonlinear filters in the TI-DFD or the optimal parameter selection. On the application side, constructing TI-DFDs for other operators such as the Radon transform or related transforms using curvelet or shearlet systems is an interesting line of future research.

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