

Nr. 86  
15. May 2022

Preprint-Series: Department of Mathematics - Applied Mathematics

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with compressed sensing data

A.Ebner, M. Haltmeier

# Convergence rates for the joint solution of inverse problems with compressed sensing data

Andrea Ebner

Department of Mathematics, University of Innsbruck  
Technikerstrasse 13, 6020 Innsbruck, Austria  
E-mail: [andrea.ebner@uibk.ac.at](mailto:andrea.ebner@uibk.ac.at)

Markus Haltmeier

Department of Mathematics, University of Innsbruck  
Technikerstrasse 13, 6020 Innsbruck, Austria  
E-mail: [markus.haltmeier@uibk.ac.at](mailto:markus.haltmeier@uibk.ac.at)

May 15, 2022

## Abstract

Compressed sensing (CS) is a powerful tool for reducing the amount of data to be collected while maintaining high spatial resolution. Such techniques work well in practice and at the same time are supported by solid theory. Standard CS results assume measurements to be made directly on the targeted signal. In many practical applications, however, CS information can only be taken from indirect data  $h_\star = \mathbf{W}x_\star$  related to the original signal by an additional forward operator. If inverting the forward operator is ill-posed, then existing CS theory is not applicable. In this paper, we address this issue and present two joint reconstruction approaches, namely relaxed  $\ell^1$  co-regularization and strict  $\ell^1$  co-regularization, for CS from indirect data. As main results, we derive error estimates for recovering  $x_\star$  and  $h_\star$ . In particular, we derive a linear convergence rate in the norm for the latter. To obtain these results, solutions are required to satisfy a source condition and the CS measurement operator is required to satisfy a restricted injectivity condition. We further show that these conditions are not only sufficient but even necessary to obtain linear convergence.

**Keywords:** Compressed sensing from indirect data, joint recovery, inverse problems, regularization, convergence rate, sparse recovery

# 1 Introduction

Compressed sensing (CS) allows to significantly reduce the amount of measurements while keeping high spatial resolution [4, 7, 9]. In mathematical terms, CS requires recovering a targeted signal  $x_\star \in \mathbb{X}$  from data  $y^\delta = \mathbf{M}x_\star + z^\delta$ . Here  $\mathbf{M}: \mathbb{X} \rightarrow \mathbb{Y}$  is the CS measurement operator,  $\mathbb{X}, \mathbb{Y}$  are Hilbert spaces and  $z^\delta \in \mathbb{Y}$  is the unknown data perturbation with  $\|z^\delta\| \leq \delta$ . CS theory shows that even when the measurement operator is severely under-determined one can derive linear error estimates  $\|x^\delta - x_\star\| = \mathcal{O}(\delta)$  for the CS reconstruction  $x^\delta$ . Such results can be derived uniformly for all sparse  $x_\star \in \mathbb{X}$  assuming the restricted isometry property (RIP) requiring that  $\|\mathbf{M}x_1 - \mathbf{M}x_2\| \asymp \|x_1 - x_2\|$  for sufficiently sparse elements [5]. The RIP is known to be satisfied with high probability for a wide range of random matrices [1]. Under a restricted injectivity condition, related results for elements satisfying a range condition are derived in [11, 12]. In [10] a strong form of the source condition has been shown to be sufficient and necessary for the uniqueness of  $\ell^1$  minimization. In [12] it is shown that the RIP implies the source condition and the restricted injectivity for all sufficiently sparse elements.

## 1.1 Problem formulation

In many applications, CS measurements can only be made on indirect data  $h_\star = \mathbf{W}x_\star$  instead of the targeted signal  $x_\star \in \mathbb{X}$ , where  $\mathbf{W}: \mathbb{X} \rightarrow \mathbb{H}$  is the forward operator coming from a specific application at hand. For example, in computed tomography, the forward operator is the Radon transform, and in microscopy, the forward operator is a convolution operator. The problem of recovering  $x_\star \in \mathbb{X}$  from CS measurements of indirect data becomes

$$\text{Recover } x_\star \text{ from } y^\delta = \mathbf{A}\mathbf{W}x_\star + z^\delta, \quad (1.1)$$

where  $\mathbf{A}: \mathbb{H} \rightarrow \mathbb{Y}$  is the CS measurement operator. In this paper we study the stable solution of (1.1).

The naive reconstruction approach is a single-step approach to consider (1.1) as standard CS problem with the composite measurement operator  $\mathbf{M} = \mathbf{A}\mathbf{W}$ . However, CS recovery conditions (such as the RIP) are not expected to hold for the composite operator  $\mathbf{A}\mathbf{W}$  due to the ill-posedness of the operator  $\mathbf{W}$ . As an alternative one may use a two-step approach where one first solves the CS problem of recovering  $\mathbf{W}x_\star$  and afterwards inverts the operator equation of the inverse problem. Apart from the additional effort, both recovery problems need to be regularized and the risk of error propagation is high. Moreover, recovering  $h_\star \in \mathbb{H}$  from sparsity alone suffers from increased non-uniqueness if  $\overline{\text{ran}(\mathbf{W})} \subsetneq \mathbb{H}$ .

## 1.2 Proposed $\ell^1$ co-regularization

In order to overcome the drawbacks of the single-step and the two-step approach, we introduce two joint reconstruction methods for solving (1.1) using a weighted  $\ell^1$  norm  $\|\cdot\|_{1,\kappa}$  (defined in (2.1)) addressing the CS part and variational regularization with an additional penalty  $\mathcal{R}$  for addressing the inverse problem part. More precisely, we study the following two regularization approaches.

- (a) STRICT  $\ell^1$  CO-REGULARIZATION: Here we construct a regularized solution pair  $(x_\alpha^\delta, h_\alpha^\delta)$  with  $h_\alpha^\delta = \mathbf{W}x_\alpha^\delta$  by minimizing

$$\mathcal{A}_{\alpha,y^\delta}(x) := \frac{1}{2}\|\mathbf{A}\mathbf{W}x - y^\delta\|^2 + \alpha(\mathcal{R}(x) + \|\mathbf{W}x\|_{1,\kappa}), \quad (1.2)$$

where  $\alpha > 0$  is a regularization parameter. This is equivalent to minimizing  $\|\mathbf{A}h - y^\delta\|^2/2 + \alpha(\mathcal{R}(x) + \|h\|_{1,\kappa})$  under the strict constraint  $h = \mathbf{W}x$ .

- (b) RELAXED  $\ell^1$  CO-REGULARIZATION: Here we relax the constraint  $h = \mathbf{W}x$  by adding a penalty and construct a regularized solution  $(x_\alpha^\delta, h_\alpha^\delta)$  by minimizing

$$\mathcal{B}_{\alpha,y^\delta}(x, h) := \frac{1}{2}\|\mathbf{W}x - h\|^2 + \frac{1}{2}\|\mathbf{A}h - y^\delta\|^2 + \alpha(\mathcal{R}(x) + \|h\|_{1,\kappa}). \quad (1.3)$$

The relaxed version in particular allows some defect between  $\mathbf{W}x_\alpha^\delta$  and  $h_\alpha^\delta$ .

Under standard assumptions, both the strict and the relaxed version provide convergent regularization methods [18].

## 1.3 Main results

As main results of this paper, under the parameter choice  $\alpha \asymp \delta$ , we derive the linear convergence rates (see Theorems 2.8, 2.7)

$$\begin{aligned} \mathcal{D}_\xi^\mathcal{R}(x_\alpha^\delta, x_\star) &= \mathcal{O}(\delta) \quad \text{as } \delta \rightarrow 0 \\ \|h_\alpha^\delta - \mathbf{W}x_\star\| &= \mathcal{O}(\delta) \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

where  $\mathcal{D}_\xi^\mathcal{R}$  is the Bregman distance with respect to  $\mathcal{R}$  and  $\xi$  (see Definition 2.1) for strict as well as for relaxed  $\ell^1$  co-regularization. In order to archive these results, we assume a restricted injectivity condition for  $\mathbf{A}$  and source conditions for  $x_\star$  and  $\mathbf{W}x_\star$ . These above error estimates are optimal in the sense that they cannot be improved even in the cases where  $\mathbf{A} = \text{Id}$ , which corresponds to an inverse problem only, or the case  $\mathbf{W} = \text{Id}$  where (1.1) is a standard CS problem on direct data.

As further main result we derive converse results, showing that the source condition and

the restricted injectivity condition are also necessary to obtain linear convergence rates (see Theorem 3.4).

We note that our results and analysis closely follow [11, 12], where the source condition and restricted injectivity are shown to be necessary and sufficient for linear convergence of  $\ell^1$ -regularization for CS on direct data. In that context, one considers CS as a particular instance of an inverse problem under a sparsity prior using variational regularization with an  $\ell^1$ -penalty (that is,  $\ell^1$ -regularization). Error estimates in the norm distance for  $\ell^1$ -regularization based on the source condition have been first derived in [14] and strengthened in [11]. In the finite dimensional setting, the source condition (under a different name) for  $\ell^1$ -regularization has been used previously in [10]. For some more recent development of  $\ell^1$ -regularization and source conditions see [8].

Further note that for the direct CS problem where  $\mathbf{W} = \text{Id}$  is the identity operator and if we take the regularizer  $\mathcal{R} = \|\cdot\|^2/2$ , then the strict  $\ell^1$  co-regularization reduces to the well known elastic net regression model [19]. Closely following the work [11], error estimates for elastic net regularization have been derived in [13]. Finally, we note that another interesting line of research in the context of  $\ell^1$  co-regularization would be the derivation of error estimates under the RIP. While we expect this to be possible, such an analysis is beyond the scope of this work.

## 2 Linear convergence rates

Throughout this paper  $\mathbb{X}, \mathbb{Y}$  and  $\mathbb{H}$  denote separable Hilbert spaces with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . Moreover we make the following assumptions.

### Assumption A.

- (A.1)  $\mathbf{W}: \mathbb{X} \rightarrow \mathbb{H}$  is linear and bounded.
- (A.2)  $\mathbf{A}: \mathbb{H} \rightarrow \mathbb{Y}$  is linear and bounded.
- (A.3)  $\mathcal{R}: \mathbb{X} \rightarrow [0, \infty]$  is proper, convex and wpsc.
- (A.4)  $\Lambda$  is a countable index set.
- (A.5)  $(\phi_\lambda)_{\lambda \in \Lambda} \in \mathbb{H}^\Lambda$  is an orthonormal basis (ONB) for  $\mathbb{H}$ .
- (A.6)  $(\kappa_\lambda)_{\lambda \in \Lambda} \in [a, \infty)^\Lambda$  for some  $a > 0$ .
- (A.7)  $\exists x \in \mathbb{X}: \mathcal{R}(x) + \sum_{\lambda \in \Lambda} \kappa_\lambda |\langle \phi_\lambda, \mathbf{W}x \rangle| < \infty$ .

Recall that  $\mathcal{R}$  is wpsc (weakly lower semi-continuous) if  $\liminf_{k \rightarrow \infty} \mathcal{R}(x_k) \geq \mathcal{R}(x)$  for all  $(x_k)_{k \in \mathbb{N}} \in \mathbb{X}^\mathbb{N}$  weakly converging to  $x \in \mathbb{X}$ . We write  $\text{ran}(\mathbf{W}) := \{\mathbf{W}x \mid x \in \mathbb{X}\}$  for the

range of  $\mathbf{W}$  and

$$\text{supp}(h) := \{\lambda \in \Lambda \mid \langle \phi_\lambda, h \rangle \neq 0\}$$

for the support of  $h \in \mathbb{H}$  with respect to  $(\phi_\lambda)_{\lambda \in \Lambda}$ . A signal  $h \in \mathbb{H}$  is sparse if  $|\text{supp}(h)| < \infty$ . The weighted  $\ell^1$ -norm  $\|\cdot\|_{1,\kappa}: \mathbb{H} \rightarrow [0, \infty]$  with weights  $(\kappa_\lambda)_{\lambda \in \Lambda}$  is defined by

$$\|h\|_{1,\kappa} := \sum_{\lambda \in \Lambda} \kappa_\lambda |\langle \phi_\lambda, h \rangle|. \quad (2.1)$$

We have  $\text{dom}(\|\cdot\|_{1,\kappa}) = \{h \in \mathbb{H} \mid \sum_{\lambda \in \Lambda} \kappa_\lambda |\langle \phi_\lambda, h \rangle| < \infty\}$ . For a finite subset of indices  $\Omega \subseteq \Lambda$ , we write

$$\mathbb{H}_\Omega := \text{span}\{\phi_\lambda \mid \lambda \in \Omega\} \quad (2.2)$$

$$\mathbf{i}_\Omega: \mathbb{H}_\Omega \rightarrow \mathbb{H}: h \mapsto h \quad (2.3)$$

$$\mathbf{A}_\Omega := \mathbf{A} \circ \mathbf{i}_\Omega: \mathbb{H}_\Omega \rightarrow \mathbb{Y} \quad (2.4)$$

$$\pi_\Omega: \mathbb{H} \rightarrow \mathbb{H}_\Omega: h \mapsto \sum_{\lambda \in \Omega} \langle \phi_\lambda, h \rangle \phi_\lambda. \quad (2.5)$$

Because  $(\phi_\lambda)_{\lambda \in \Lambda} \subseteq \mathbb{H}$  is an ONB, every  $h \in \mathbb{H}$  has the basis representation  $h = \sum_{\lambda \in \Lambda} \langle \phi_\lambda, h \rangle \phi_\lambda$ . Finally,  $\|\mathbf{A}\|$  denotes the standard operator norm.

## 2.1 Auxiliary estimates

One main ingredient for our results are error estimates for general variational regularization in terms of the Bregman distance. Recall that  $\xi \in \mathbb{X}$  is called subgradient of a functional  $\mathcal{Q}: \mathbb{X} \rightarrow [0, \infty]$  at  $x_\star \in \mathbb{X}$  if

$$\forall x \in \mathbb{X}: \quad \mathcal{Q}(x) \geq \mathcal{Q}(x_\star) + \langle \xi, x - x_\star \rangle.$$

The set of all subgradients is called the subdifferential of  $\mathcal{Q}$  at  $x_\star$  and denoted by  $\partial\mathcal{Q}(x_\star)$ .

**Definition 2.1** (Bregman distance). Given  $\mathcal{Q}: \mathbb{X} \rightarrow [0, \infty]$  and  $\xi \in \partial\mathcal{Q}(x_\star)$ , the Bregman distance between  $x_\star, x \in \mathbb{X}$  with respect to  $\mathcal{Q}$  and  $\xi$  is defined by

$$\mathcal{D}_\xi^\mathcal{Q}(x, x_\star) := \mathcal{Q}(x) - \mathcal{Q}(x_\star) - \langle \xi, x - x_\star \rangle. \quad (2.6)$$

The Bregman distance is a valuable tool for deriving error estimates for variational regularization. Specifically, for our purpose we use the following convergence rates result.

**Lemma 2.2** (Variational regularization). *Let  $\mathbf{M}: \mathbb{X} \rightarrow \mathbb{Y}$  be bounded and linear, let  $\mathcal{Q}: \mathbb{X} \rightarrow [0, \infty]$  be proper, convex and wpsc and let  $(x_\star, y_\star) \in \mathbb{X} \times \mathbb{Y}$  satisfy  $\mathbf{M}x_\star = y_\star$  and  $\mathbf{M}^*\eta \in \partial\mathcal{Q}(x_\star)$  for some  $\eta \in \mathbb{Y}$ . Then for all  $\delta, \alpha > 0$ ,  $y^\delta \in \mathbb{Y}$  with  $\|y^\delta - y_\star\| \leq \delta$  and*

$x_\alpha^\delta \in \operatorname{argmin}\{\|\mathbf{M}x - y^\delta\|^2/2 + \alpha\mathcal{Q}(x)\}$  we have

$$\|\mathbf{M}x_\alpha^\delta - y^\delta\| \leq \delta + 2\alpha\|\eta\| \quad (2.7)$$

$$D_{\mathbf{M}^*\eta}^{\mathcal{Q}}(x_\alpha^\delta, x_\star) \leq (\delta + \alpha\|\eta\|)^2/(2\alpha). \quad (2.8)$$

*Proof.* Lemma 2.2 has been derived in [12, Lemma 3.5]. Note that error estimates for variational regularization in the Bregman distance have first been derived in [3].  $\square$

For our purpose we will apply Lemma 2.2 where  $\mathcal{Q}$  is a combination formed by  $\mathcal{R}$  and  $\|\cdot\|_{1,\kappa}$ . We will use that the subdifferential of  $\|\cdot\|_{1,\kappa}$  at  $h_\star$  consists of all  $\eta = \sum_{\lambda \in \Lambda} \eta_\lambda \phi_\lambda \in \mathbb{H}$  with

$$\begin{cases} \eta_\lambda = \kappa_\lambda \operatorname{sign}(\langle \phi_\lambda, h_\star \rangle) & \text{for } \lambda \in \operatorname{supp}(h_\star) \\ \eta_\lambda \in [-\kappa_\lambda, \kappa_\lambda] & \text{for } \lambda \notin \operatorname{supp}(h_\star). \end{cases}$$

Since the family  $(\eta_\lambda)_{\lambda \in \Lambda}$  is square summable,  $\eta_\lambda = \pm\kappa_\lambda$  can be obtained for only finitely many  $\lambda$  and therefore  $\partial\|h_\star\|_{1,\kappa}$  is nonempty if and only if  $h_\star$  is sparse.

**Remark 2.3** (Weighted  $\ell^1$ -norm). For  $\eta = \sum_{\lambda \in \Lambda} \eta_\lambda \phi_\lambda \in \partial\|h_\star\|_{1,\kappa}$  define

$$\Omega[\eta] := \{\lambda \in \Lambda : |\eta_\lambda| = \kappa_\lambda\} \quad (2.9)$$

$$m[\eta] := \min\{\kappa_\lambda - |\eta_\lambda| : \lambda \notin \Omega[\eta]\}. \quad (2.10)$$

Then  $\Omega[\eta]$  is finite and as  $(\eta_\lambda)_{\lambda \in \Lambda}$  converges to zero,  $m[\eta]$  is well-defined with  $m[\eta] > 0$ . Because  $\|\cdot\|_{1,\kappa}$  is positively homogeneous it holds  $\|h_\star\|_1 = \langle \eta, h_\star \rangle$ . Thus, for  $h \in \mathbb{H}$ ,

$$\begin{aligned} \mathcal{D}_\eta^{\|\cdot\|_{1,\kappa}}(h, h_\star) &= \|h\|_{1,\kappa} - \langle \eta, h \rangle \\ &= \sum_{\lambda \in \Lambda} (\kappa_\lambda |\langle \phi_\lambda, h \rangle| - \eta_\lambda \langle \phi_\lambda, h \rangle) \\ &\geq \sum_{\lambda \in \Lambda} (\kappa_\lambda - |\eta_\lambda|) |\langle \phi_\lambda, h \rangle| \\ &\geq m[\eta] \sum_{\lambda \notin \Omega[\eta]} |\langle \phi_\lambda, h \rangle|. \end{aligned} \quad (2.11)$$

Estimate (2.11) implies that if  $\mathcal{D}_\eta^{\|\cdot\|_{1,\kappa}}(h_\alpha^\delta, h_\star)$  linearly converge to 0, then so does  $\sum_{\lambda \notin \Omega[\eta]} |\langle \phi_\lambda, h_\alpha^\delta \rangle|$ .

**Lemma 2.4.** Let  $\Omega \subseteq \Lambda$  be finite,  $\mathbf{A}_\Omega: \mathbb{H}_\Omega \rightarrow \mathbb{Y}$  injective and  $h_\star \in \mathbb{H}_\Omega$ . Then, for all  $h \in \mathbb{H}$ ,

$$\|h - h_\star\| \leq \|\mathbf{A}_\Omega^{-1}\| \|\mathbf{A}h - \mathbf{A}h_\star\| + (1 + \|\mathbf{A}_\Omega^{-1}\| \|\mathbf{A}\|) \sum_{\lambda \notin \Omega} |\langle \phi_\lambda, h \rangle|. \quad (2.12)$$

*Proof.* Because  $\mathbb{H}_\Omega$  is finite dimensional and  $\mathbf{A}_\Omega$  is injective, the inverse  $\mathbf{A}_\Omega^{-1}: \text{ran}(\mathbf{A}_\Omega) \rightarrow \mathbb{H}_\Omega$  is well defined and bounded. Consequently,

$$\begin{aligned} \|h - h_\star\| &\leq \|\pi_\Omega h - h_\star\| + \|\pi_{\Lambda \setminus \Omega} h\| \\ &\leq \|\mathbf{A}_\Omega^{-1}\| \|\mathbf{A}_\Omega(\pi_\Omega h - h_\star)\| + \|\pi_{\Lambda \setminus \Omega} h\| \\ &\leq \|\mathbf{A}_\Omega^{-1}\| \|\mathbf{A}(h - h_\star) - \mathbf{A}\pi_{\Lambda \setminus \Omega} h\| + \|\pi_{\Lambda \setminus \Omega} h\| \\ &\leq \|\mathbf{A}_\Omega^{-1}\| \|\mathbf{A}h - \mathbf{A}h_\star\| + (1 + \|\mathbf{A}_\Omega^{-1}\| \|\mathbf{A}\|) \|\pi_{\Lambda \setminus \Omega} h\|. \end{aligned}$$

Bounding the  $\ell^2$ -norm by the  $\ell^1$ -norm yields (2.12).  $\square$

**Lemma 2.5.** *Let  $h_\star \in \mathbb{H}$  be sparse,  $\eta \in \partial\|h_\star\|_{1,\kappa}$  and assume that  $\mathbf{A}_{\Omega[\eta]}: \mathbb{H}_{\Omega[\eta]} \rightarrow \mathbb{Y}$  is injective. Then, for  $h \in \mathbb{H}$ ,*

$$\|h - h_\star\| \leq \|\mathbf{A}_{\Omega[\eta]}^{-1}\| \|\mathbf{A}h - \mathbf{A}h_\star\| + \frac{1 + \|\mathbf{A}_{\Omega[\eta]}^{-1}\| \|\mathbf{A}\|}{m[\eta]} \mathcal{D}_\eta^{\|\cdot\|_{1,\kappa}}(h, h_\star). \quad (2.13)$$

*Proof.* Follows from (2.12), (2.11).  $\square$

## 2.2 Relaxed $\ell^1$ co-regularization

First we derive linear rates for the relaxed model  $\mathcal{B}_{\alpha,y^\delta}$ . These results will be derived under the following condition.

**Condition 1.**

$$(1.1) \quad (x_\star, h_\star, y_\star) \in \mathbb{X} \times \mathbb{H} \times \mathbb{Y} \text{ with } \mathbf{W}x_\star = h_\star, \mathbf{A}h_\star = y_\star.$$

$$(1.2) \quad \exists u \in \mathbb{H}: \mathbf{W}^*u \in \partial\mathcal{R}(x_\star)$$

$$(1.3) \quad \exists v \in \mathbb{Y}: \mathbf{A}^*v - u \in \partial\|h_\star\|_{1,\kappa}$$

$$(1.4) \quad \mathbf{A}_{\Omega[\mathbf{A}^*v - u]} \text{ is injective.}$$

Conditions (1.2), (1.3) are source conditions very commonly assumed in regularization theory. From (1.3) it follows that  $h_\star$  is sparse and contained in  $\mathbb{H}_{\Omega[\mathbf{A}^*v - u]}$ . Condition (1.4) is the restricted injectivity condition.

**Remark 2.6** (Product formulation). We introduce the operator  $\mathbf{M}: \mathbb{X} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{Y}$  and the functional  $\mathcal{Q}: \mathbb{X} \times \mathbb{H} \rightarrow [0, \infty]$ ,

$$\mathbf{M}(x, h) := (\mathbf{W}x - h, \mathbf{A}h) \quad (2.14)$$

$$\mathcal{Q}(x, h) := \mathcal{R}(x) + \|h\|_{1,\kappa}. \quad (2.15)$$



Using these notions, one can rewrite the relaxed co-regularization functional  $\mathcal{B}_{\alpha, y^\delta}$  as

$$\mathcal{B}_{\alpha, y^\delta}(x, h) = \frac{1}{2} \|\mathbf{M}(x, h) - (0, y^\delta)\|^2 + \alpha \mathcal{Q}(x, h).$$

Because  $\mathbf{W}$  and  $\mathbf{A}$  are linear and bounded,  $\mathbf{M}$  is linear and bounded, too. Moreover, since  $\mathcal{R}$  and  $\|\cdot\|_{1, \kappa}$  are proper, convex and wsc,  $\mathcal{Q}$  has these properties, too. The subdifferential  $\partial \mathcal{Q}(x_\star, h_\star)$  is given by  $\partial \mathcal{Q}(x_\star, h_\star) = \partial \mathcal{R}(x_\star) \times \partial \|h_\star\|_{1, \kappa}$ . The Bregman distance with respect to  $\xi = (\xi_1, \xi_2)$  is given by

$$\mathcal{D}_{(\xi_1, \xi_2)}^{\mathcal{Q}}((x, h), (x_\star, h_\star)) = \mathcal{D}_{\xi_1}^{\mathcal{R}}(x - x_\star) + \mathcal{D}_{\xi_2}^{\|\cdot\|_{1, \kappa}}(h - h_\star). \quad (2.16)$$

(1.2), (1.3) can be written as  $\mathbf{M}^*(u, v) \in \partial \mathcal{Q}(x_\star, h_\star)$ .

Here comes our main estimate for the relaxed model.

**Theorem 2.7** (Relaxed  $\ell^1$  co-regularization). *Let Condition 1 hold and consider the parameter choice  $\alpha = C\delta$  for  $C > 0$ . Then for all  $y^\delta \in \mathbb{Y}$  with  $\|y^\delta - y_\star\| \leq \delta$  and all  $(x_\alpha^\delta, h_\alpha^\delta) \in \operatorname{argmin} \mathcal{B}_{\alpha, y^\delta}$  we have*

$$\mathcal{D}_{\mathbf{W}^*u}^{\mathcal{R}}(x_\alpha^\delta, x_\star) \leq c_{(u, v)} \delta, \quad (2.17)$$

$$\|h_\alpha^\delta - h_\star\| \leq d_{(u, v)} \delta, \quad (2.18)$$

where

$$c_{(u, v)} := (1 + C\|(u, v)\|)^2 / (2C)$$

$$d_{(u, v)} := 2\|\mathbf{A}_{\Omega[\mathbf{A}^*v-u]}^{-1}\|(1 + C\|(u, v)\|) + \frac{1 + \|\mathbf{A}_{\Omega[\mathbf{A}^*v-u]}^{-1}\|\|\mathbf{A}\|}{m[\eta]} c_{(u, v)}.$$

*Proof.* According to (1.3),  $\eta := \mathbf{A}^*v - u \in \partial \|h_\star\|_{1, \kappa}$ , which implies that  $\Omega[\eta]$  is finite and  $h_\star \in \mathbb{H}_{\Omega[\eta]}$ . With (1.4) and Lemma 2.5 we therefore get

$$\|h_\alpha^\delta - h_\star\| \leq \|\mathbf{A}_{\Omega[\eta]}^{-1}\|\|\mathbf{A}h_\alpha^\delta - y_\star\| + \frac{(1 + \|\mathbf{A}_{\Omega[\eta]}^{-1}\|\|\mathbf{A}\|)}{m[\eta]} \mathcal{D}_\eta^{\|\cdot\|_{1, \kappa}}(h_\alpha^\delta, h_\star). \quad (2.19)$$

Using the product formulation as in Remark 2.6, according to (1.2), (1.3) the source condition  $\mathbf{M}^*(u, v) \in \partial \mathcal{Q}(x_\star, h_\star)$  holds. By Lemma 2.2 and the choice  $\alpha = C\delta$  we obtain

$$\begin{aligned} \|\mathbf{M}(x_\alpha^\delta, h_\alpha^\delta) - (0, y^\delta)\| &\leq (1 + 2C\|(u, v)\|) \delta \\ \mathcal{D}_{\mathbf{M}^*(u, v)}^{\mathcal{Q}}((x_\alpha^\delta, h_\alpha^\delta), (x_\star, h_\star)) &\leq (1 + C\|(u, v)\|)^2 / (2C) \delta. \end{aligned}$$

Using (2.14), (2.15), (2.16) we obtain

$$\|\mathbf{A}h_\alpha^\delta - y^\delta\| \leq (1 + 2C\|(u, v)\|) \delta$$

$$\begin{aligned}
\|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| &\leq (1 + 2C\|(u, v)\|) \delta \\
\mathcal{D}_\eta^{\|\cdot\|_{1,\kappa}}(h_\alpha^\delta, h_\star) &\leq (1 + C\|(u, v)\|)^2 \delta / (2C) \\
\mathcal{D}_{\mathbf{W}^*u}^{\mathcal{R}}(x_\alpha^\delta, x_\star) &\leq (1 + C\|(u, v)\|)^2 \delta / (2C).
\end{aligned}$$

Combining this with (2.19) completes the proof.  $\square$

If  $\mathcal{R}$  is totally convex, then convergence in the Bregman distance implies convergence in the norm [18, Lemma 3.31]. For example, for the standard penalty  $\mathcal{R} = \|\cdot\|^2/2$  from Theorem 2.7 one deduces the rate  $\|x_\alpha^\delta - x_\star\| = \mathcal{O}(\sqrt{\delta})$ .

### 2.3 Strict $\ell^1$ co-regularization

Next we analyze the strict approach (1.2). We derive linear convergence rates under the following condition.

**Condition 2.**

$$(2.1) \quad (x_\star, y_\star) \in \mathbb{X} \times \mathbb{Y} \text{ satisfies } \mathbf{A}\mathbf{W}x_\star = y_\star.$$

$$(2.2) \quad \exists \nu \in \mathbb{Y}: \mathbf{W}^*\mathbf{A}^*\nu \in \partial(\mathcal{R} + \partial\|\mathbf{W}(\cdot)\|_{1,\kappa})(x_\star)$$

$$(2.3) \quad \exists \xi \in \partial\mathcal{R}(x_\star) \exists \eta \in \partial\|\cdot\|_1(\mathbf{W}x_\star): \mathbf{W}^*\mathbf{A}^*\nu = \xi + \mathbf{W}^*\eta$$

$$(2.4) \quad \mathbf{A}_{\Omega[\eta]} \text{ is injective.}$$

Condition (2.2) is a source condition for the forward operator  $\mathbf{W}\mathbf{A}$  and the regularization functional  $\mathcal{R} + \partial\|\mathbf{W}(\cdot)\|_{1,\kappa}$ . Condition (2.3) assumes the splitting of the subgradient  $\mathbf{W}^*\mathbf{A}^*\nu = \xi + \mathbf{W}^*\eta$  into subgradients  $\xi \in \partial\mathcal{R}(x_\star)$  and  $\mathbf{W}^*\eta \in \partial\|\mathbf{W}(\cdot)\|_1(x_\star)$ . The assumption (2.4) is the restricted injectivity.

**Theorem 2.8** (Strict  $\ell^1$  co-regularization). *Let Condition 2 hold and consider the parameter choice  $\alpha = C\delta$  for  $C > 0$ . Then for  $\|y^\delta - y_\star\| \leq \delta$  and  $x_\alpha^\delta \in \operatorname{argmin} \mathcal{A}_{\alpha, y^\delta}$  we have*

$$\mathcal{D}_\xi^{\mathcal{R}}(x_\alpha^\delta, x_\star) \leq c_{(\nu, \eta)} \delta \tag{2.20}$$

$$\|\mathbf{W}x_\alpha^\delta - \mathbf{W}x_\star\| \leq d_{(\nu, \eta)} \delta, \tag{2.21}$$

with the constants

$$\begin{aligned}
c_{(\nu, \eta)} &:= (1 + C\|\nu\|)^2 / (2C) \\
d_{(\nu, \eta)} &:= 2\|\mathbf{A}_{\Omega[\eta]}^{-1}\| (1 + C\|\nu\|) + \frac{1 + \|\mathbf{A}_{\Omega[\eta]}^{-1}\| \|\mathbf{A}\|}{m[\eta]} c_{(\nu, \eta)}.
\end{aligned}$$

*Proof.* Condition 2 implies that  $\Omega[\eta]$  finite and  $\mathbf{W}x_\star \in \mathbb{H}_{\Omega[\eta]}$ . From Lemma 2.5 we obtain

$$\|\mathbf{W}x_\alpha^\delta - \mathbf{W}x_\star\| \leq \|\mathbf{A}_{\Omega[\eta]}^{-1}\| \|\mathbf{A}\mathbf{W}x_\alpha^\delta - y_\star\| + \frac{(1 + \|\mathbf{A}_{\Omega[\eta]}^{-1}\| \|\mathbf{A}\|)}{m[\eta]} \mathcal{D}_\eta^{\|\cdot\|_1}(\mathbf{W}x_\alpha^\delta, \mathbf{W}x_\star). \quad (2.22)$$

According to (2.2) and Lemma 2.2 applied with  $\mathbf{M} = \mathbf{A}\mathbf{W}$  and  $\mathcal{Q} = \mathcal{R} + \|\mathbf{W}(\cdot)\|_{1,\kappa}$  we obtain

$$\|\mathbf{A}\mathbf{W}x_\alpha^\delta - y^\delta\| \leq (1 + 2C\|\nu\|) \delta \quad (2.23)$$

$$\mathcal{D}_{\mathbf{W}^\star \mathbf{A}^\star \nu}^{\mathcal{Q}}(x_\alpha^\delta, x_\star) \leq (1 + C\|\nu\|)^2 / (2C) \delta. \quad (2.24)$$

From (2.2) we obtain  $\mathcal{D}_{\mathbf{W}^\star \mathbf{A}^\star \nu}^{\mathcal{Q}} = \mathcal{D}_\eta^{\|\cdot\|_1}(\mathbf{W}(\cdot), \mathbf{W}(\cdot)) + \mathcal{D}_\xi^{\mathcal{R}}$ . Together with (2.22), (2.23), (2.23) this show the claim.  $\square$

### 3 Necessary Conditions

In this section we show that the source condition and restricted injectivity are not only sufficient but also necessary for linear convergence of relaxed  $\ell^1$  co-regularization. In the following we restrict ourselves to the  $\ell^1$ -norm

$$\|\cdot\|_1 := \|\cdot\|_{1,1} = \sum_{\lambda \in \Lambda} |\langle \phi_\lambda, \cdot \rangle|.$$

We denote by  $\mathbf{M}$  and  $\mathcal{Q}$  the product operator and regularizer defined in (2.14), (2.15). We call  $(x_\star, h_\star)$  a  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$  if  $x_\star \in \operatorname{argmin}\{\mathcal{Q}(x) \mid \mathbf{M}(x, h) = (0, y_\star)\}$ . In this section we fix the following list of assumptions which is slightly stronger than Assumption A.

#### Assumption B.

(B.1)  $\mathbf{W}: \mathbb{X} \rightarrow \mathbb{H}$  is linear and bounded with dense range.

(B.2)  $\mathbf{A}: \mathbb{H} \rightarrow \mathbb{Y}$  is linear and bounded.

(B.3)  $\mathcal{R}: \mathbb{H} \rightarrow [0, \infty]$  is proper, strictly convex and wpsc.

(B.4)  $\Lambda$  is countable index set.

(B.5)  $(\phi_\lambda)_{\lambda \in \Lambda} \subseteq \mathbb{H}$  is an ONB of  $\mathbb{H}$ .

(B.6)  $\forall \lambda \in \Lambda: \phi_\lambda \in \operatorname{ran}(\mathbf{W})$ .

(B.7)  $\exists x \in \mathbb{X}: \mathcal{R}(x) + \|\mathbf{W}x\|_1 < \infty$ .

(B.8)  $\mathcal{R}$  is Gateaux differentiable at  $x_\star$  if  $(x_\star, h_\star)$  is the unique  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$ .

Under Assumption B, the equation  $\mathbf{M}(x, h) = (0, y_\star)$  has a unique  $\mathcal{Q}$ -minimizing solution.

**Condition 3.**

$$(3.1) \quad (x_\star, h_\star, y_\star) \in \mathbb{X} \times \mathbb{H} \times \mathbb{Y} \text{ with } \mathbf{A}h_\star = y_\star, \mathbf{W}x_\star = h_\star.$$

$$(3.2) \quad \exists u \in \mathbb{H}: \mathbf{W}^*u \in \partial\mathcal{R}(x_\star)$$

$$(3.3) \quad \exists v \in \mathbb{Y}: \mathbf{A}^*v - u \in \partial\|h_\star\|_1$$

$$(3.4) \quad \forall \lambda \notin \text{supp}(h_\star): |\langle \phi_\lambda, \mathbf{A}^*v - u \rangle| < 1$$

$$(3.5) \quad \mathbf{A}_{\text{supp}[h_\star]} \text{ is injective.}$$

**3.1 Auxiliary results**

Condition 3 is clearly stronger than Condition 1. Below we will show that these conditions are actually equivalent. For that purpose we start with several lemmas. These results are in the spirit of [12] where necessary conditions for standard  $\ell^1$  minimization have been derived.

**Lemma 3.1.** *Assume that  $(x_\star, h_\star) \in \mathbb{X} \times \mathbb{H}$  is the unique  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$ , let  $u \in \mathbb{H}$  satisfy  $\mathbf{W}^*u = \partial\mathcal{R}(x_\star)$  and assume that  $h_\star$  is sparse. Then:*

(a) *The restricted mapping  $\mathbf{A}_{\text{supp}(h_\star)}$  is injective.*

(b) *For every finite set  $\Omega_1$  with  $\text{supp}(h_\star) \cap \Omega_1 = \emptyset$  there exists  $\theta \in \mathbb{Y}$  such that*

$$\begin{aligned} \forall \lambda \in \text{supp}(h_\star): \langle \phi_\lambda, \mathbf{A}^*\theta - u \rangle &= \text{sign}(\langle \phi_\lambda, h_\star \rangle) \\ \forall \lambda \in \Omega_1: |\langle \phi_\lambda, \mathbf{A}^*\theta - u \rangle| &< 1. \end{aligned}$$

*Proof.* (a): Denote  $\Omega := \text{supp}(h_\star)$ . After possibly replacing some basis vectors by  $-\phi_\lambda$ , we may assume without loss of generality that  $\text{sign}(\langle \phi_\lambda, h_\star \rangle) = 1$  for  $\lambda \in \Omega$ . Since  $(x_\star, h_\star)$  is the unique  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$ , it follows that

$$\mathcal{Q}(x_\star, h_\star) < \mathcal{Q}(x_\star + tx, h_\star + t\mathbf{W}x)$$

for all  $t \neq 0$  and all  $x \in \mathbb{X}$  with  $w := \mathbf{W}x \in \ker(\mathbf{A}) \setminus \{0\}$ . Because  $\Omega$  is finite, the mapping

$$t \mapsto \|h_\star + tw\|_1 = \sum_{\lambda \in \Omega} |\langle \phi_\lambda, h_\star \rangle + t\langle \phi_\lambda, w \rangle| + |t| \sum_{\lambda \notin \Omega} |\langle \phi_\lambda, w \rangle|$$

is piecewise linear. Taking the one-sided directional derivative of  $\mathcal{Q}$  with respect to  $t$ , we have

$$\begin{aligned}
0 &< \lim_{t \downarrow 0} \frac{\mathcal{Q}(x_\star + tx, h_\star + tw) - \mathcal{Q}(x_\star, h_\star)}{t} \\
&= \lim_{t \downarrow 0} \frac{\|h_\star + tw\|_1 - \|h_\star\|_1}{t} + \lim_{t \downarrow 0} \frac{\mathcal{R}(x_\star + tx) - \mathcal{R}(x_\star)}{t} \\
&= \sum_{\lambda \in \Omega} \langle \phi_\lambda, w \rangle + \sum_{\lambda \notin \Omega} |\langle \phi_\lambda, w \rangle| + \langle \mathbf{W}^*u, x \rangle.
\end{aligned} \tag{3.1}$$

For the last equality we used that  $\langle \phi_\lambda, h_\star \rangle = 1$  for all  $\lambda \in \Omega$ , that  $\mathcal{R}$  is Gateaux differentiable and that  $\mathbf{W}^*u = \partial \mathcal{R}(x_\star)$ . Inserting  $-(x, w)$  instead of  $(x, w)$  in (3.1) we deduce

$$\sum_{\lambda \notin \Omega} |\langle \phi_\lambda, w \rangle| > \left| \sum_{\lambda \in \Omega} \langle \phi_\lambda, w \rangle + \langle u, w \rangle \right| \tag{3.2}$$

for all  $w \in (\ker(\mathbf{A}) \cap \text{ran}(\mathbf{W})) \setminus \{0\}$ . In particular,

$$\forall w \in (\ker(\mathbf{A}) \cap \text{ran}(\mathbf{W})) \setminus \{0\}: \sum_{\lambda \notin \Omega} |\langle \phi_\lambda, w \rangle| > 0 \tag{3.3}$$

and consequentially  $\ker(\mathbf{A}) \cap \text{ran}(\mathbf{W}) \cap \mathbb{H}_\Omega = \{0\}$ . Because  $\phi_\lambda \in \text{ran}(\mathbf{W})$  for all  $\lambda \in \Lambda$  and  $\Omega$  is finite, we have  $\mathbb{H}_\Omega \subseteq \text{ran}(\mathbf{W})$ . Therefore  $\ker(\mathbf{A}) \cap \mathbb{H}_\Omega = \{0\}$  which verifies that  $\mathbf{A}_\Omega$  is injective.

(b): Let  $\Omega_1 \subseteq \Lambda$  be finite with  $\Omega \cap \Omega_1 = \emptyset$ . Inequality (3.2) and the finiteness of  $\Omega \cup \Omega_1$  imply the existence of a constant  $\mu \in (0, 1)$  such that, for  $w \in \ker(\mathbf{A}) \cap \mathbb{H}_{\Omega \cup \Omega_1}$ ,

$$\mu \sum_{\lambda \in \Omega_1} |\langle \phi_\lambda, w \rangle| \geq \left| \sum_{\lambda \in \Omega} \langle \phi_\lambda, w \rangle + \langle u, w \rangle \right|. \tag{3.4}$$

Assume for the moment  $\xi \in \text{ran}(\mathbf{A}_{\Omega \cup \Omega_1}^*)$ . Then  $\xi = \mathbf{A}_{\Omega \cup \Omega_1}^* \theta$  for some  $\theta \in \mathbb{Y}$ . Due to (B.5),  $\pi_\Omega$  is an orthogonal projection and the adjoint of the embedding  $i_\Omega$ . The identity  $\pi_{\Omega \cup \Omega_1} \circ \mathbf{A}^* = (\mathbf{A} \circ i_{\Omega \cup \Omega_1})^* = \mathbf{A}_{\Omega \cup \Omega_1}^*$  implies that

$$\forall \lambda \in \Omega \cup \Omega_1: \langle \phi_\lambda, \xi \rangle = \langle \phi_\lambda, \mathbf{A}_{\Omega \cup \Omega_1}^* \theta \rangle = \langle \phi_\lambda, \mathbf{A}^* \theta \rangle.$$

By assumption,  $\mathbb{H}_{\Omega \cup \Omega_1}$  is finite dimensional and therefore  $\text{ran}(\mathbf{A}_{\Omega \cup \Omega_1}^*) = \ker(\mathbf{A}_{\Omega \cup \Omega_1})^\perp \subseteq \mathbb{H}_{\Omega \cup \Omega_1}$ , where  $(\cdot)^\perp$  denotes the orthogonal complement in  $\mathbb{H}_{\Omega \cup \Omega_1}$ . Consequently we have to show the existence of  $\xi \in (\ker(\mathbf{A}_{\Omega \cup \Omega_1}))^\perp \subseteq \mathbb{H}_{\Omega \cup \Omega_1}$  with

$$\begin{aligned}
\langle \phi_\lambda, \xi \rangle &= 1 + u_\lambda & \forall \lambda \in \Omega, \\
\langle \phi_\lambda, \xi \rangle &\in (u_\lambda - 1, u_\lambda + 1) & \forall \lambda \in \Omega_1,
\end{aligned} \tag{3.5}$$

where  $u_\lambda := \langle \phi_\lambda, u \rangle$ .

Define the element  $z \in \mathbb{H}_{\Omega \cup \Omega_1}$  by  $\langle \phi_\lambda, z \rangle = 1 + u_\lambda$  for  $\lambda \in \Omega$  and  $\langle \phi_\lambda, z \rangle = u_\lambda$  for  $\lambda \in \Omega_1$ . If  $z \in (\ker(\mathbf{A}))^\perp$ , then we choose  $\xi := z$  and (3.5) is fulfilled. If, on the other hand,  $z \notin (\ker(\mathbf{A}))^\perp$ , then  $\dim(\ker(\mathbf{A}_{\Omega \cup \Omega_1})) =: s \geq 1$  and there exists a basis  $(w^{(1)}, \dots, w^{(s)})$  of  $\ker(\mathbf{A}_{\Omega \cup \Omega_1})$  such that

$$\begin{aligned}
1 &= \langle z, w^{(i)} \rangle \\
&= \sum_{\lambda \in \Omega} (1 + u_\lambda) \langle \phi_\lambda, w^{(i)} \rangle + \sum_{\lambda \in \Omega_1} u_\lambda \langle \phi_\lambda, w^{(i)} \rangle \\
&= \sum_{\lambda \in \Omega} \langle \phi_\lambda, w^{(i)} \rangle + \sum_{\lambda \in \Omega \cup \Omega_1} u_\lambda \langle \phi_\lambda, w^{(i)} \rangle + \sum_{\lambda \notin \Omega \cup \Omega_1} u_\lambda \langle \phi_\lambda, w^{(i)} \rangle \\
&= \sum_{\lambda \in \Omega} \langle \phi_\lambda, w^{(i)} \rangle + \langle u, w^{(i)} \rangle \quad \forall i \in \{1, \dots, s\}
\end{aligned} \tag{3.6}$$

Consider now the constrained minimization problem on  $\mathbb{H}_{\Omega_1}$

$$\begin{aligned}
&\max_{\lambda \in \Omega_1} |\langle \phi_\lambda, z' \rangle| \rightarrow \min \\
&\text{subject to } \langle z', w^{(i)} \rangle = -1 \quad \text{for } i \in \{1, \dots, s\}.
\end{aligned} \tag{3.7}$$

Because of the equality  $1 = \langle z, w^{(i)} \rangle$ , the admissible vectors  $z'$  in (3.7) are precisely those for which  $\xi := z + z' \in (\ker(\mathbf{A}_{\Omega \cup \Omega_1}))^\perp$ . Thus, the task of finding  $\xi$  satisfying (3.5) reduces to showing that the value of (3.7) is strictly smaller than 1. Note that the dual of the convex function  $z' \mapsto \max_{\lambda \in \Omega_1} |\langle \phi_\lambda, z' \rangle|$  is the function

$$\max_{\Omega_1} \ni z' \mapsto \begin{cases} 0 & \text{if } \sum_{\lambda \in \Omega_1} |\langle \phi_\lambda, z' \rangle| \leq 1 \\ +\infty & \text{if } \sum_{\lambda \in \Omega_1} |\langle \phi_\lambda, z' \rangle| > 1. \end{cases} \tag{3.8}$$

Recalling that  $\langle z', w^{(i)} \rangle = \sum_{\lambda \in \Omega_1} \langle \phi_\lambda, z' \rangle \phi_\lambda w^{(i)}$ , it follows that the dual problem to (3.7) is the following constrained problem on  $\mathbb{R}^s$ :

$$\begin{aligned}
S(p) &:= - \sum_{i=1}^s p_i \rightarrow \min \\
&\text{subject to } \sum_{\lambda \in \Omega_1} \left| \sum_{i=1}^s p_i \langle \phi_\lambda, w^{(i)} \rangle \right| \leq 1.
\end{aligned} \tag{3.9}$$

From (3.6) we obtain that

$$\sum_{\lambda \in \Omega} \sum_{i=1}^s p_i \langle \phi_\lambda, w^{(i)} \rangle + \sum_{i=1}^s p_i \langle u, w^{(i)} \rangle = \sum_{i=1}^s p_i = -S(p)$$

for every  $p \in \mathbb{R}^s$ . Taking  $w = \sum_{i=1}^s p_i w^{(i)} \in \ker(\mathbf{A}) \cap \mathbb{H}_{\Omega \cup \Omega_1}$ , inequality (3.4) therefore

implies that for every  $p \in \mathbb{R}^s$  there exist  $\mu \in (0, 1)$  such that

$$\begin{aligned} & \mu \sum_{\lambda \in \Omega_1} \left| \sum_{i=1}^s p_i \langle \phi_\lambda, w^{(i)} \rangle \right| \\ & \geq \left| \sum_{\lambda \in \Omega} \sum_{i=1}^s p_i \langle \phi_\lambda, w^{(i)} \rangle + \sum_{i=1}^s p_i \langle u, w^{(i)} \rangle \right| \\ & = \left| \sum_{i=1}^s p_i \right| = |S(p)|. \end{aligned}$$

From (3.9) it follows that  $|S(p)| \leq \mu$  for every admissible vector  $p \in \mathbb{R}^s$  for (3.9). Thus the value of  $S(p)$  in (3.9) is greater than or equal to  $-\mu$ . Since the value of the primal problem (3.7) is the negative of the dual problem (3.9), this shows that the value of (3.7) is at most  $\mu$ . As  $\mu \in (0, 1)$ , this proves that the value of (3.7) is strictly smaller than 1 and, as we have shown above, this proves assertion (3.5).  $\square$

**Lemma 3.2.** *Assume that  $(x_\star, h_\star) \in \mathbb{X} \times \mathbb{H}$  is the unique  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$  and suppose  $\mathbf{W}^*u \in \partial\mathcal{R}(x_\star)$ ,  $\mathbf{A}^*v - u \in \partial\|h_\star\|_1$  for some  $(u, v) \in \mathbb{H} \times \mathbb{Y}$ . Then  $(x_\star, h_\star)$  satisfies Condition 3.*

*Proof.* The restricted injectivity condition (3.5) follows from Lemma 3.1. Conditions (3.1), (3.2) are satisfied according to assumption. Define now

$$\begin{aligned} \Omega_1 &:= \Omega[\mathbf{A}^*v - u] \setminus \text{supp}(h_\star) \\ &= \{\lambda \in \Lambda \setminus \text{supp}(h_\star) \mid |\langle \phi_\lambda, \mathbf{A}^*v - u \rangle| = 1\}. \end{aligned}$$

Because  $(\langle \phi_\lambda, \mathbf{A}^*v - u \rangle)_{\lambda \in \Lambda} \subseteq \ell^2(\Lambda)$ , the set  $\Omega_1$  is finite. Let  $\theta \in \mathbb{Y}$  be as in Lemma 3.1 (b) and set

$$\begin{aligned} \|\theta\|_\infty &:= \sup\{|\langle \phi_\lambda, \mathbf{A}^*\theta - u \rangle| \mid \lambda \in \Lambda\} \\ a &:= (1 - m[\mathbf{A}^*v - u]) / (2\|\theta\|_\infty) \\ \hat{v} &:= (1 - a)v + a\theta. \end{aligned}$$

Note that  $a \in (0, 1/2]$ . Then the following hold:

- If  $\lambda \in \text{supp}(h_\star)$ , then

$$\langle \phi_\lambda, \mathbf{A}^*\hat{v} - u \rangle = (1 - a)\langle \phi_\lambda, \mathbf{A}^*v - u \rangle + a\langle \phi_\lambda, \mathbf{A}^*\theta - u \rangle = \text{sign}(\langle \phi_\lambda, h_\star \rangle).$$

- If  $\lambda \in \Omega_1$ , then

$$|\langle \phi_\lambda, \mathbf{A}^*\hat{v} - u \rangle| \leq (1 - a)|\langle \phi_\lambda, \mathbf{A}^*v - u \rangle| + a|\langle \phi_\lambda, \mathbf{A}^*\theta - u \rangle| < (1 - a) + a = 1.$$

- If  $\lambda \in \Lambda \setminus (\text{supp}(h_\star) \cup \Omega_1)$ , then

$$\begin{aligned}
& | \langle \phi_\lambda, \mathbf{A}^* \hat{v} - u \rangle | \\
& \leq (1 - a) m[\mathbf{A}^* v - u] + a \|\theta\|_\infty \\
& \leq m[\mathbf{A}^* v - u] + (1 - m[\mathbf{A}^* v - u])/2 \\
& = (1 + m[\mathbf{A}^* v - u])/2 < 1.
\end{aligned}$$

Consequently  $(u, \hat{v})$  satisfies (3.3), (3.4).  $\square$

**Lemma 3.3.** *Let  $(\delta_k)_{k \in \mathbb{N}} \in (0, \infty)^\mathbb{N}$  converge to 0,  $(y_k)_{k \in \mathbb{N}} \in \mathbb{Y}^\mathbb{N}$  satisfy  $\|y_k - y_\star\| \leq \delta_k$  and  $x_k \in \text{argmin } \mathcal{B}_{\alpha, y^\delta}$  with  $\alpha_k \geq C\delta_k$  for  $C > 0$ . Then  $\|x_k - x_\star\| \rightarrow 0$  and  $\|\mathbf{M}x_k - y_k\| = \mathcal{O}(\delta_k)$  as  $k \rightarrow \infty$  imply  $\text{ran}(\mathbf{M}^*) \cap \partial\mathcal{Q}(x_\star) \neq \emptyset$ .*

*Proof.* See [12, Lemma 4.1]. The proof given there also applies to our situation.  $\square$

### 3.2 Main result

The following theorem in the main results of this section and shows that the source condition and restricted injectivity are in fact necessary for linear convergence.

**Theorem 3.4** (Converse results). *Let  $(x_\star, h_\star, y_\star) \in \mathbb{X} \times \mathbb{H} \in \mathbb{Y}$  satisfy  $\mathbf{A}h_\star = y_\star$  and  $\mathbf{W}x_\star = h_\star$  and let Assumption 3 hold. Then the following statements are equivalent:*

- (i)  $(x_\star, h_\star, y_\star)$  satisfies Condition 3.
- (ii)  $(x_\star, h_\star, y_\star)$  satisfies Condition 1.
- (iii)  $\exists \xi \in \partial\mathcal{R}(x_\star) \forall C > 0 \exists c_1, c_2 > 0$ : For  $\alpha = C\delta$ ,  $\|y^\delta - y_\star\| \leq \delta$ , and  $(x_\alpha^\delta, h_\alpha^\delta) \in \text{argmin } \mathcal{B}_{\alpha, y^\delta}$  we have

$$\mathcal{D}_\xi^{\mathcal{R}}(x_\alpha^\delta, x_\star) \leq c_1 \delta \tag{3.10}$$

$$\|h_\alpha^\delta - h_\star\| \leq c_2 \delta. \tag{3.11}$$

- (iv)  $(x_\star, h_\star)$  is the unique  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$  and  $\forall C > 0 \exists c_3, c_4 > 0$  with

$$\|\mathbf{A}h_\alpha^\delta - y_\star\| \leq c_3 \delta \tag{3.12}$$

$$\|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| \leq c_4 \delta \tag{3.13}$$

for  $\|y - y_\star\| \leq \delta$ ,  $\alpha = C\delta$ ,  $(x_\alpha^\delta, h_\alpha^\delta) \in \text{argmin } \mathcal{B}_{\alpha, y^\delta}$ .



*Proof.* Item (i) obviously implies Item (ii). The implication (ii)  $\Rightarrow$  (iii) has been shown in Theorem 2.7. The rate in (iii) implies that  $h_\star$  is the second component of every  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$ . As  $\mathcal{R}$  is strictly convex, (3.10) implies that  $(x_\star, h_\star)$  is the unique  $\mathcal{Q}$ -minimizing solution of  $\mathbf{M}(x, h) = (0, y_\star)$ . The rate in (3.12) follows trivially from (3.11), since  $\mathbf{A}$  is linear and bounded. To prove (3.13) we choose  $u \in \mathbb{H}$  with  $\partial\mathcal{R}(x_\star) = \mathbf{W}^*u$  and proceed similar as in the proof of Lemma 2.2. Because  $(x_\alpha^\delta, h_\alpha^\delta) \in \operatorname{argmin} \mathcal{B}_{\alpha, y^\delta}$ ,

$$\begin{aligned} \|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\|^2 + \|\mathbf{A}h_\alpha^\delta - y^\delta\|^2 + 2\alpha\mathcal{R}(x_\alpha^\delta) + 2\alpha\|h_\alpha^\delta\|_1 \\ \leq \|h_\star - h_\alpha^\delta\|^2 + \|\mathbf{A}h_\alpha^\delta - y^\delta\|^2 + 2\alpha\mathcal{R}(x_\star) + 2\alpha\|h_\alpha^\delta\|_1 \end{aligned}$$

and therefore

$$\|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\|^2 \leq (c_2\delta)^2 + 2\alpha(\mathcal{R}(x_\star) - \mathcal{R}(x_\alpha^\delta)).$$

By the definition of the Bregman distance

$$\begin{aligned} \mathcal{R}(x_\star) - \mathcal{R}(x_\alpha^\delta) \\ \leq -D_{\mathbf{W}^*u}^{\mathcal{R}}(x_\alpha^\delta, x_\star) - \langle u, \mathbf{W}x_\alpha^\delta - h_\star \rangle \\ \leq -D_{\mathbf{W}^*u}^{\mathcal{R}}(x_\alpha^\delta, x_\star) + \|u\| \|\mathbf{W}x_\alpha^\delta - h_\star\| \\ \leq -D_{\mathbf{W}^*u}^{\mathcal{R}}(x_\alpha^\delta, x_\star) + \|u\| \|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| + \|u\|c_2\delta. \end{aligned}$$

Since the Bregman distance is nonnegative, it follows

$$\begin{aligned} 0 &\geq \|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\|^2 - 2\alpha\|u\| \|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| - 2\alpha\|u\|c_2\delta - (c_2\delta)^2 \\ &= \left( \|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| + c_2\delta \right) \cdot \left( \|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| - 2\alpha\|u\| - c_2\delta \right) \end{aligned}$$

and hence  $\|\mathbf{W}x_\alpha^\delta - h_\alpha^\delta\| \leq (2C\|u\| + c_2)\delta$ .

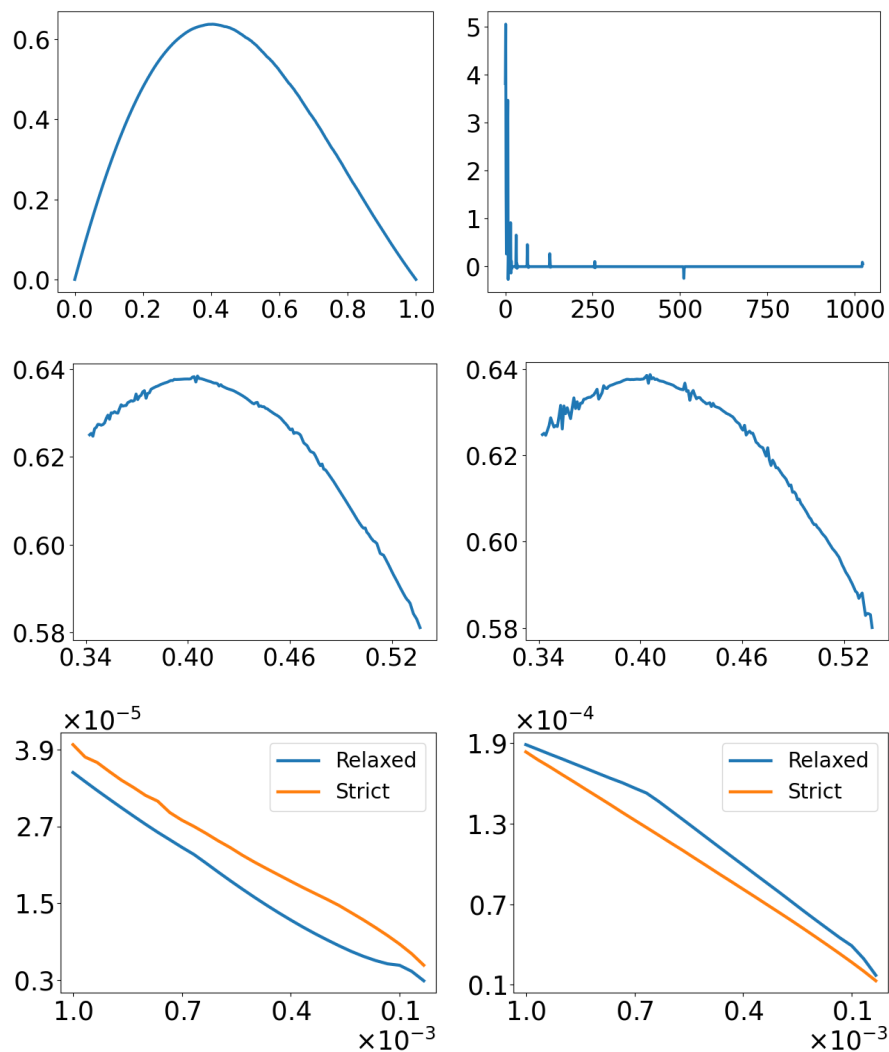
Now let (iv) hold and  $\|y - y_k\| \leq \delta_k$ ,  $\delta_k \rightarrow 0$ . Choose  $\alpha_k = C\delta_k$  and  $(x_k, h_k) \in \operatorname{argmin} \mathcal{B}_{\alpha_k, y_k}$ . The uniqueness of  $(x_\star, h_\star)$  implies  $\|(x_k, h_k) - (x_\star, h_\star)\| \rightarrow 0$  as  $k \rightarrow \infty$ , see [11, Prop. 7]. Moreover,

$$\|(\mathbf{W}x_k - h_k, \mathbf{A}h_k - y_k)\| \leq \|\mathbf{W}x_\alpha^\delta - h_k\| + \|\mathbf{A}h_k - y_k\| \leq (c_4 + c_3 + 1)\delta_k.$$

Lemma 3.3 implies  $\operatorname{ran}(\mathbf{M}^*) \cap \partial\mathcal{R}(x_\star, h_\star) \neq \emptyset$ , which means that there exists  $(u, v) \in \mathbb{H} \times \mathbb{Y}$  such that  $\mathbf{W}^*u \in \partial\mathcal{R}(x_\star)$  and  $\mathbf{A}^*v - u \in \partial\|h_\star\|_1$ . Proposition 3.2 finally shows that Condition 3 holds, which concludes the proof.  $\square$

### 3.3 Numerical example

We consider recovering a function from CS measurements of its primitive. The aim of this elementary example is to point out possible implementation of the two proposed models and supporting the linear error estimates. Detailed comparison with other methods and figuring out limitations of each method is subject of future research.



**Figure 1:** Top:  $x_*$  (left) and  $\mathbf{W}x_*$  (right). Middle: Reconstruction using relaxed (left) and strict (right)  $\ell^1$  co-regularization. Bottom:  $\|x_* - x_\alpha^\delta\|^2/2$  (left) and  $\|h_* - h_\alpha^\delta\|$  (right) as a functions of noise level  $\delta$ .

The discrete operator  $\mathbf{W}: \mathbb{R}^N \rightarrow \mathbb{R}^N$  is taken as a discretization of the integration operator  $L^2[0, 1] \rightarrow L^2[0, 1]: f \mapsto \int_0^t f$ . The CS measurement matrix  $\mathbf{A}: \mathbb{R}^{m \times N}$  is taken as random Bernoulli matrix with entries 0, 1. We apply strict and relaxed  $\ell^1$  co-regularization with  $\mathcal{R} = \|\cdot\|^2/2$ ,  $(\phi_\lambda)_{\lambda \in \Lambda}$  as Daubechies wavelet ONB with two vanishing moments and  $\kappa_\lambda = 1$ . For minimizing the relaxed  $\ell^1$  co-regularization functional we apply

the Douglas-Rachford algorithm [6, Algorithm 4.2] and for strict  $\ell^1$  co-regularization we apply the ADMM algorithm [6, Algorithm 6.4] applied to the constraint formulation  $\operatorname{argmin}_{x,h} \{ \|\mathbf{A}\mathbf{W}x - y^\delta\|^2/2 + \alpha\|x\|^2/2 + \alpha\|h\|_1 \mid \mathbf{W}x = h \}$ .

Results are shown in Figure 1. The top row depicts the targeted signal  $x_\star \in \mathbb{R}^N$  (left) for which  $\mathbf{W}x_\star$  is sparsely represented by  $(\phi_\lambda)_{\lambda \in \Lambda}$  (right). The middle row shows reconstructions using the strict and the relaxed co-sparse regularization from noisy data  $\|y - y_\star\| \leq 10^{-5}$ . The bottom row plots  $\mathcal{D}_{x_\star}^{\mathcal{R}}(x_\alpha^\delta, x_\star)$  and  $\|h_\alpha^\delta - \mathbf{W}x_\star\|$  as functions of the noise level. Note that for  $\mathcal{R} = \|\cdot\|^2/2$  the Bregman distance is given by  $\mathcal{D}_{x_\star}^{\mathcal{R}}(x_\alpha^\delta, x_\star) = \|x_\alpha^\delta - x_\star\|^2/2$ . Both error plots show a linear convergence rate supporting Theorems 2.7, 2.8, 3.4.

## 4 Conclusion

While the theory of CS on direct data is well developed, this by far not the case when compressed measurements are made on indirect data. For that purpose, in this paper we study CS from indirect data written as composite problem  $y^\delta = \mathbf{A}\mathbf{W}x_\star + z^\delta$  where  $\mathbf{A}$  models the CS measurement operator and  $\mathbf{W}$  the forward model generating indirect data and depending on the application at hand. For signal reconstruction we have proposed two novel reconstruction methods, named relaxed and strict  $\ell^1$  co-regularization, for jointly estimating  $x$  and  $h_\star = \mathbf{A}x$ . Note that the main conceptual difference between the proposed method over standard CS is that we use the  $\ell^1$  penalty for indirect data  $\mathbf{W}x_\star$  instead of  $x_\star$  together with another penalty for  $x_\star$  accounting for the inversion of  $\mathbf{A}$ , and jointly recovering both unknowns.

As main results for both reconstruction models we derive linear error estimates under source conditions and restricted injectivity (see Theorems 2.8, 2.7). Moreover, conditions have been shown to be even necessary to obtain such results (see Theorem 3.4). Our results have been illustrated on a simple numerical example for combined CS and numerical differentiation. In future work further detailed numerical investigations are in order comparing our models with standard CS approaches in practical important applications demonstrating strengths and limitations of different methods. Potential applications include magnetic resonance imaging [2, 15] or photoacoustic tomography [17, 16].

## Acknowledgment

The presented work of A.E. and M.H. has been supported of the Austrian Science Fund (FWF), project P 30747-N32.

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