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Regularization of Inverse Problems by Filtered Diagonal Frame Decomposition

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Abstract

The characteristic feature of inverse problems is their instability with respect to data perturbations. In order to stabilize the inversion process, regularization methods have to be developed and applied. In this work we introduce and analyze the concept of filtered diagonal frame decomposition which extends the standard filtered singular value decomposition to the frame case. Frames as generalized singular system allows to better adapt to a given class of potential solutions. In this paper, we show that filtered diagonal frame decomposition yield a convergent regularization method. Moreover, we derive convergence rates under source type conditions and prove order optimality under the assumption that the considered frame is a Riesz-basis.
keywords Inverse problems, frame decomposition, convergence analysis, convergence rates.

1 Introduction

This paper is concerned with solving inverse problems of the form

$$y^\delta = \mathbf{K}\, x + z\,, \tag{1}$$

where $\mathbf{K}\colon \mathbb{X} \to \mathbb{Y}$ is a bounded linear operator between Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$, and z denotes the data distortion that satisfies $\|z\| \leq \delta$ for some noise level $\delta \geq 0$. Many inverse problems can be written in the form (1). A main characteristic property of inverse problems is that they are ill-posed [7, 15]. This means that the solution of (1) is either not unique or is unstable with respect to data perturbations of the right-hand side.

Arguably, the theory of solving inverse problems of the form (1) is quite well developed. Especially, the class of filter based methods gives a wide range of such methods. Assuming

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that $\mathbf{K}$ has a singular value decomposition $\mathbf{K} = \sum_{n\in\mathbb{N}} \sigma_n \langle \cdot, u_n \rangle v_n$, these methods have one of the following equivalent forms

$$\mathbf{F}_\alpha(y^\delta) = \sum_{n\in\mathbb{N}} h_\alpha(\sigma_n^2) \langle \mathbf{K}^* y^\delta, u_n \rangle u_n \tag{2}$$

$$\mathbf{F}_\alpha(y^\delta) = \sum_{n\in\mathbb{N}} g_\alpha(\sigma_n) \langle y^\delta, v_n \rangle u_n \,. \tag{3}$$

Here $(h_\alpha)_{\alpha>0}$ is a family functions converging pointwise to $1/\lambda$ as $\alpha \to 0$ and $g_\alpha(\sigma) := \sigma h_\alpha(\sigma^2)$. In particular, $g_\alpha(\sigma) \to 1/\sigma$ as $\alpha \to 0$. The expression (3) can be interpreted as regularized version of the SVD based formula

$$\forall y \in \operatorname{dom}(\mathbf{K}^\dagger)\colon \qquad \mathbf{K}^\dagger\, y = \sum_{n\in\mathbb{N}} \frac{1}{\sigma_n} \langle y, v_n \rangle u_n \tag{4}$$

for the pseudo inverse $\mathbf{K}^\dagger$. The analysis of such regularization methods can be found, for example, in [7, 12]

The SVD cannot be adapted to the underlying signal class and therefore is not always a good representation for various kinds of inverse problems. Instead, certain diagonal frame decompositions generalizing the SVD are better suited because the defining frames can be adjusted to a particular application [1, 5, 9]. To the best of our knowledge, filter based methods based on diagonal frame decompositions have not been rigorously studied in the context of regularization theory. This paper addresses this issue and develops a regularization theory for diagonalizing systems including the SVD based filter methods as special case.

1.1 Filtered diagonal frame decomposition

A diagonal frame decomposition (DFD) for the operator $\mathbf{K}$ consists of a frame $(u_\lambda)_{\lambda\in\Lambda}$ of $(\ker \mathbf{K})^\perp$, a frame $(v_\lambda)_{\lambda\in\Lambda}$ of $\overline{\operatorname{ran} \mathbf{K}}$ and a sequence of positive numbers $(\kappa_\lambda)_{\lambda\in\Lambda}$ such that the pseudo inverse has the form

$$\mathbf{K}^\dagger\, y = \sum_{\lambda\in\Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda. \tag{5}$$

Here $(\bar{u}_\lambda)_\lambda$ is a dual frame of $(u_\lambda)_{\lambda\in\Lambda}$ and $\kappa_\lambda > 0$ are generalized singular values. It is therefore a generalization of the SVD allowing frames as non-orthogonal generalized singular systems $(u_\lambda)_\lambda$ and $(v_\lambda)_\lambda$. Moreover, both systems are in general overcomplete, which is one of the main reasons for using frames. Opposed to the SVD, many different DFDs for a given operator can exist and can be adapted to the particular signal class.

In the case of ill-posed problems where $\mathbf{K}^\dagger$ is unbounded, regularization techniques have to be applied. Based on a DFD, in this paper, we consider the filtered DFD

$$\mathbf{F}_\alpha(y^\delta) := \sum_{\lambda\in\Lambda} f_\alpha(\kappa_\lambda) \langle y^\delta, v_\lambda \rangle \bar{u}_\lambda \,.$$

Here $(f_\alpha)_{\alpha>0}$ is a family of functions converging pointwise to $1/\kappa$ as $\alpha \to 0$. In case we take the SVD as the DFD then the filtered DFD reduces to classical filter based regularization.

However, the filtered DFD contains other interesting special cases. In particular, taking $(u_\lambda)_{\lambda\in\Lambda}$ as wavelet, curvelet and shearlet system yields DFDs for image reconstruction [1, 2, 5, 9]. We also point out that such systems are used in variational regularization schemes [3, 4, 6, 10, 11, 13, 14] which are related but different from the approach followed in this paper.

1.2 Outline

In Section 2 we introduce and study the concept of a DFD. We show that the DFD yields an associated inversion formula and relate the ill-posedness of the inverse problem (1) to the decay of the quasi-singular values. In Section 3 we introduce the filtered DFD to account the ill-posedness. We show that the filtered DFD provides a regularization method and we derive convergence rates under source-type conditions. The paper concludes with a short discussion and outlook given in Section 4.

2 Operator inversion by diagonal frame decomposition

Throughout this paper $\mathbb{X}$ and $\mathbb{Y}$ denote Hilbert spaces and $\mathbf{K}\colon \operatorname{dom}(\mathbf{K}) \subseteq \mathbb{X} \to \mathbb{Y}$ a linear operator. In this section, we introduce diagonal frame compositions which in the following sections will be used to regularise the inverse problem defined by $\mathbf{K}$.

2.1 Frames

We start by briefly recalling some basic facts about frame theory. A family $\boldsymbol{u} = (u_\lambda)_{\lambda\in\Lambda} \in \mathbb{U}^\Lambda$ where Λ is an at most countable index set is called frame for the Hilbert space $\mathbb{U}$ if there are constants $A, B > 0$ such that

$$\forall x \in \mathbb{U}\colon \quad A\|x\|^2 \leq \sum_{\lambda\in\Lambda} |\langle u_\lambda, x\rangle|^2 \leq B\|x\|^2 \,. \tag{6}$$

The constants A and B are called lower an upper frame bounds of $\boldsymbol{u}$. The frame is called tight if $A = B$ and exact if it fails to be a frame whenever any single element is deleted from the sequence $(u_\lambda)_{\lambda\in\Lambda}$. A frame that is not a Riesz basis is said to be overcomplete.

Definition 1 (Analysis and Synthesis Operator)**.** Let $\boldsymbol{u} = (u_\lambda)_{\lambda\in\Lambda}$ be a frame for the Hilbert space $\mathbb{U}$. The analysis and synthesis operator of $\boldsymbol{u}$, respectively, are defined by

$$\mathbf{T}_{\boldsymbol{u}}\colon \mathbb{U} \to \ell^2(\Lambda) : x \mapsto (\langle x, u_\lambda\rangle)_{\lambda\in\Lambda} \tag{7}$$

$$\mathbf{T}_{\boldsymbol{u}}^*\colon \ell^2(\Lambda) \to \mathbb{X}\colon (c_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} c_\lambda u_\lambda \,. \tag{8}$$

One easily verifies that $\mathbf{T}_{\boldsymbol{u}}$ and $\mathbf{T}_{\boldsymbol{u}}^*$ are linear bounded operators and the synthesis operator $\mathbf{T}_{\boldsymbol{u}}^*$ is the adjoint of the analysis operator $\mathbf{T}_{\boldsymbol{u}}$.

Definition 2 (Dual Frame)**.** Let $\boldsymbol{u} = (u_\lambda)_{\lambda\in\Lambda}$ be a frame for the Hilbert space $\mathbb{U}$. A frame $\bar{\boldsymbol{u}} = (\bar{u}_\lambda)_{\lambda\in\Lambda}$ for $\mathbb{U}$ is called a dual frame of $\boldsymbol{u}$ if the following duality condition holds:

$$\forall x \in \mathbb{U}\colon \quad x = \sum_{\lambda\in\Lambda} \langle x, u_\lambda\rangle \bar{u}_\lambda = \mathbf{T}_{\bar{\boldsymbol{u}}}^* \, \mathbf{T}_{\boldsymbol{u}} \, x \,. \tag{9}$$

Every frame has at least one dual frame and if the frame $\boldsymbol{u}$ is over-complete, then there exist infinitely many dual frames of $\boldsymbol{u}$.

Definition 3 (Norm bounded Frames)**.** Let $\mathbb{U}$ be a Hilbert space and $\boldsymbol{u}$ a frame for $\mathbb{U}$. We call $\boldsymbol{u}$ norm bounded from below if there exists a constant $a > 0$ such that $\inf_{\lambda\in\Lambda} \|u_\lambda\| \geq a$.

Note that every frame is already norm bounded from above. In fact, the upper frame condition implies $\|u_\lambda\|^4 = |\langle u_\lambda, u_\lambda\rangle|^2 \leq \sum_{\mu\in\Lambda} |\langle u_\lambda, u_\mu\rangle|^2 \leq B\|u_\lambda\|^2$ which gives $\sup_{\lambda\in\Lambda} \|u_\lambda\| \leq \sqrt{B}$. On the other hand one easily constructs examples of frames that are not norm bounded from below.

2.2 Diagonal frame decomposition

We use the following notion extending the wavelet-vaguelette decomposition (WVD) and biorthogonal curvelet decomposition to more general frames. It will allow us to unify and extend existing filter based regularization methods.

Definition 4 (Diagonal Frame Decomposition, DFD)**.** Let $\mathbf{K}\colon \operatorname{dom}(\mathbf{K}) \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear operator and Λ an at most countable index set. We call $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa}) = (u_\lambda, v_\lambda, \kappa_\lambda)_{\lambda\in\Lambda}$ a diagonal frame decomposition (DFD) for the operator $\mathbf{K}$ if the following holds:

(D1) $(u_\lambda)_{\lambda\in\Lambda}$ is a frame for $(\ker \mathbf{K})^\perp \subseteq \mathbb{X}$,

(D2) $(v_\lambda)_{\lambda\in\Lambda}$ is a frame for $\overline{\operatorname{ran} \mathbf{K}} \subseteq \mathbb{Y}$,

(D3) $(\kappa_\lambda)_{\lambda\in\Lambda} \in (0,\infty)^\Lambda$ satisfies the quasi-singular relations

$$\forall \lambda \in \Lambda\colon \quad \mathbf{K}^* \, v_\lambda = \kappa_\lambda u_\lambda \,. \tag{10}$$

We call $(\kappa_\lambda)_{\lambda\in\Lambda}$ quasi-singular values and $(u_\lambda)_{\lambda\in\Lambda}$, $(v_\lambda)_{\lambda\in\Lambda}$ corresponding quasi-singular system.

In the case $\boldsymbol{u}$ is an orthonormal wavelet basis, then the DFD reduces the WVD introduced in [5]. A WVD decomposition has been constructed for the classical computed tomography modeled by the two-dimensional Radon transform see [5]. In the case of the two-dimensional Radon transform, a biorthogonal curvelet decomposition was constructed in [1]. In [2], the authors derived biorthogonal shearlet decompositions for two- and three-dimensional Radon transforms. The limited data case has been studied in [8].

Remark 5. *Let $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ be a DFD for $\mathbf{K}$ and, for any sequence $\boldsymbol{a} = (a_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}^\Lambda$, set* $\operatorname{dom}(\mathbf{M}_{\boldsymbol{a}}) := \{(c_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda) \mid (a_\lambda c_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda)\}$ *and define the diagonal operator*

$$\mathbf{M}_{\boldsymbol{a}}\colon \operatorname{dom}(\mathbf{M}_{\boldsymbol{a}}) \subseteq \ell^2(\Lambda) \to \ell^2(\Lambda)\colon (c_\lambda)_{\lambda\in\Lambda} \mapsto (a_\lambda c_\lambda)_{\lambda\in\Lambda} \,. \tag{11}$$

In particular, for the quasi singular values $\boldsymbol{\kappa}$ we have $\operatorname{ran} \mathbf{T}_{\boldsymbol{u}} \subseteq \operatorname{dom}(\mathbf{M}_{\boldsymbol{\kappa}})$. *Moreover,* (10) *is equivalent to $\forall x \in \mathbb{X}\ \forall \lambda \in \Lambda\colon\ \langle \mathbf{K}^* \, v_\lambda, x\rangle = \kappa_\lambda \, \langle u_\lambda, x\rangle$ which in turn shows that the quasi singular value relation* (10) *is equivalent to the identity* $\mathbf{T}_{\boldsymbol{v}} \, \mathbf{K} = \mathbf{M}_\kappa \, \mathbf{T}_{\boldsymbol{u}}$.

Theorem 6 (Inversion formula via DFD)**.** *Let* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ *a DFD for* $\mathbf{K}$ *and* $\bar{\boldsymbol{u}} = (\bar{u}_\lambda)_{\lambda\in\Lambda}$ *a dual frame of* $\boldsymbol{u}$*. Then we have*

$$\forall y \in \operatorname{dom}(\mathbf{K}^\dagger)\colon \qquad \mathbf{K}^\dagger\, y = \sum_{\lambda\in\Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda. \tag{12}$$

In particular, we have $\mathbf{K}^\dagger = \mathbf{T}_{\bar{\boldsymbol{u}}}^* \, \mathbf{M}_{1/\boldsymbol{\kappa}} \, \mathbf{T}_{\boldsymbol{v}}$ *where* $1/\boldsymbol{\kappa}$ *denotes the pointwise inverse of* $\boldsymbol{\kappa}$*.*

Proof. For every element $y \in \operatorname{dom}(\mathbf{K}^\dagger) = \operatorname{ran}(\mathbf{K}) \oplus \operatorname{ran}(\mathbf{K})^\perp$ define $\mathbf{B}\, y := \sum_{\lambda\in\Lambda} \kappa_\lambda^{-1} \langle y, v_\lambda \rangle \bar{u}_\lambda$. We will show that the mapping $\mathbf{B}\colon\ \operatorname{dom}(\mathbf{K}^\dagger) \subseteq \mathbb{Y} \to \mathbb{X}\colon y \mapsto \mathbf{B}\, y$ equals the Moore-Penrose inverse. For that purpose note that any element in $\operatorname{dom}(\mathbf{K}^\dagger)$ has the unique representation $y = \mathbf{K}\, x^\dagger + y^\perp$ where $x^\dagger \in \ker(\mathbf{K})^\perp$ and $y^\perp \in \operatorname{ran}(\mathbf{K})^\perp$. We have

$$\begin{aligned} \mathbf{B}\, y = \sum_{\lambda\in\Lambda} \frac{1}{\kappa_\lambda} \langle y, v_\lambda \rangle \bar{u}_\lambda = \sum_{\lambda\in\Lambda} \frac{1}{\kappa_\lambda} \langle \mathbf{K}\, x^\dagger, v_\lambda \rangle \bar{u}_\lambda = \sum_{\lambda\in\Lambda} \frac{1}{\kappa_\lambda} \langle x^\dagger, \mathbf{K}^*\, v_\lambda \rangle \bar{u}_\lambda \\ = \sum_{\lambda\in\Lambda} \langle x^\dagger, u_\lambda \rangle \bar{u}_\lambda = x^\dagger = \mathbf{K}^\dagger\, y\,. \end{aligned} \tag{13}$$

Here we used the definition of $\mathbf{B}$, the fact that $v_\lambda \in \operatorname{ran}(\mathbf{K})$, the quasi-singular relation (10), and the fact that $\bar{\boldsymbol{u}}$ is a dual frame of $\boldsymbol{u}$ for $(\ker \mathbf{K})^\perp$. □

2.3 Ill-posedness and quasi singular values

Many inverse problems are unstable in the sense that the Moore-Penrose inverse is unbounded. It is well known that the Moore-Penrose inverse of an operator having a SVD is bounded if and only if the singular values do not accumulate at zero. Below we show that a similar characterization holds for the quasi-singular values in a DFD.

Theorem 7 (Characterization of ill-posedness via DFD)**.** *Let* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ *be a DFD of* $\mathbf{K}$*. Then the following assertions hold.*

(a) If $\inf_{\lambda\in\Lambda} \kappa_\lambda > 0$*, then* $\mathbf{K}^\dagger$ *is bounded.*

(b) Suppose $\boldsymbol{v}$ *is norm bounded from below. Then the converse implication holds*

$$\mathbf{K}^\dagger \ \textit{is bounded} \ \Rightarrow \inf_{\lambda\in\Lambda} \kappa_\lambda > 0.$$

Proof. (a) Let $\bar{\boldsymbol{u}}$ be a dual frame of $\boldsymbol{u}$. Then, for every $y \in \operatorname{dom}(\mathbf{K}^\dagger)$ we have

$$\begin{aligned} \left\|\mathbf{K}^\dagger\, y\right\|^2 = \left\| \sum_{\lambda\in\Lambda} \kappa_\lambda^{-1} \, \langle y, v_\lambda \rangle \, \bar{u}_\lambda \right\|^2 \leq \|\, \mathbf{T}_{\bar{\boldsymbol{u}}}^* \,\|^2 \sum_{\lambda\in\Lambda} \left| \kappa_\lambda^{-1} \, \langle y, v_\lambda \rangle \right|^2 \\ \leq \frac{\|\, \mathbf{T}_{\bar{\boldsymbol{u}}}^* \,\|^2}{(\inf_{\lambda\in\Lambda} \kappa_\lambda)^2} \sum_{\lambda\in\Lambda} |\langle y, v_\lambda \rangle|^2 \leq \frac{\|\, \mathbf{T}_{\bar{\boldsymbol{u}}}^* \,\|^2 \|\, \mathbf{T}_{\boldsymbol{v}} \,\|^2}{(\inf_{\lambda\in\Lambda} \kappa_\lambda)^2} \|y\|^2, \end{aligned}$$

which implies $\mathbf{K}^\dagger$ is bounded.

(b) Let $\mathbf{K}^\dagger$ be bounded with bound $\| \mathbf{K}^\dagger \|$ and suppose that $\inf_{\lambda\in\Lambda} \kappa_\lambda = 0$. Then the family $(\kappa_\lambda^{-1} v_\lambda)_{\lambda\in\Lambda}$ has no upper frame bound. This can be shown by contradiction: Suppose it has an upper frame B we know that $\sup_{\lambda\in\Lambda} \|\kappa_\lambda^{-1} v_\lambda\| \leq \sqrt{B}$, but since $\boldsymbol{v}$ is norm bounded from below we have $\sup_{\lambda\in\Lambda} \|\kappa_\lambda^{-1} v_\lambda\| = \infty$. Hence we have that for all constants $B > 0$ there exists $y \in \overline{\operatorname{ran} \mathbf{K}}$ such that

$$\sum_{\lambda\in\Lambda} \left|\left\langle y, \kappa_\lambda^{-1} v_\lambda \right\rangle\right|^2 > B\|y\|^2. \tag{14}$$

Now choose $B = \| \mathbf{T}_{\bar{\boldsymbol{u}}}^\dagger \|^2 \| \mathbf{K}^\dagger \|^2$, where $\bar{\boldsymbol{u}}$ is an arbitrary dual frame of $\boldsymbol{u}$, and y such that (14) is satisfied. It is well known that if $\mathbf{K}^\dagger$ is bounded, $\mathbf{K}$ has closed range. Thereby, $y \in \operatorname{dom}(\mathbf{K}^\dagger)$. Moreover, it has the unique representation $y = \mathbf{K}\, x^\dagger$ with $x^\dagger \in (\ker \mathbf{K})^\perp$ and by $\left\langle \mathbf{K}\, x^\dagger, \kappa_\lambda^{-1} v_\lambda \right\rangle = \left\langle x^\dagger, u_\lambda \right\rangle$ follows that $(\left\langle y, \kappa_\lambda^{-1} v_\lambda \right\rangle)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$. Then we have

$$\begin{aligned} \left\|\mathbf{K}^\dagger\, y\right\|^2 = \left\|\sum_{\lambda\in\Lambda} \kappa_\lambda^{-1} \left\langle y, v_\lambda \right\rangle \bar{u}_\lambda\right\|^2 &\geq \frac{1}{\| \mathbf{T}_{\bar{\boldsymbol{u}}}^\dagger \|^2} \sum_{\lambda\in\Lambda} \left|\left\langle y, \kappa_\lambda^{-1} v_\lambda \right\rangle\right|^2 \\ &> \frac{1}{\| \mathbf{T}_{\bar{\boldsymbol{u}}}^\dagger \|^2} B\|y\|^2 = \| \mathbf{K}^\dagger \|^2 \|y\|^2 , \end{aligned}$$

which leads to a contradiction. □

Compact operators with infinite dimensional range are typical examples of linear operators with non-closed range. Moreover, the spectral theorem for compact operators states that zero is the only accumulation point of the singular values $(\sigma_\lambda)_{\lambda\in\Lambda}$. This means that we can find a bijection $\pi\colon \mathbb{N} \to \Lambda$ such that $(\kappa_{\pi(n)})_{n\in\mathbb{N}}$ is a decreasing null-sequence. Below we show that the same holds for the DFD if $\boldsymbol{u}$ is norm bounded from below.

Theorem 8 (Quasi-singular values for compact operators)**.** *Suppose that* $\mathbf{K}\colon \mathbb{X} \to \mathbb{Y}$ *is a compact linear operator and assume that* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ *is a DFD for* $\mathbf{K}$*, where* $\boldsymbol{u}$ *is norm bounded from below. Then, zero is the only accumulation point of the family* $\boldsymbol{\kappa}$ *of quasi-singular values.*

Proof. Without loss of generality consider the case $\Lambda = \mathbb{N}$. Aiming for a contradiction, we assume that $\boldsymbol{\kappa}$ has an accumulation point different from zero (∞ is allowed). Therefore we can find a subsequence $(\kappa_{n(k)})_{k\in\mathbb{N}}$ with $\inf_{k\in\mathbb{N}} \kappa_{n(k)} := c > 0$. Consequently $\|v_{n(k)}/\kappa_{n(k)}\| \leq c^{-1}\sqrt{B_{\boldsymbol{v}}}$, where $B_{\boldsymbol{v}}$ is the upper frame bound of $\boldsymbol{v}$. In particular, the sequence $(v_{n(k)}/\kappa_{n(k)})_{k\in\mathbb{N}}$ is bounded. Because $\mathbf{K}^*$ is compact, there exists another subsequence $(v_{n(k(\ell))}/\kappa_{n(k(\ell))})_{k\in\mathbb{N}}$ such that $u_{n(k(\ell))} = \mathbf{K}^*(v_{n(k(\ell))}/\kappa_{n(k(\ell))})$ strongly converges to some $x \in \operatorname{ran} \mathbf{K}^* = (\ker \mathbf{K})^\perp$. Because $\boldsymbol{u}$ is norm bounded from below we have $x \neq 0$. Choose $\epsilon > 0$ such that $\|x\|^2 \geq 2\epsilon$. Since $u_{n(k(\ell))} \to x$ we can choose $N \in \mathbb{N}$ such that $\forall \ell \geq N\colon \|u_{n(k(\ell))} - x\|^2 < \epsilon$. From this it follows $2\left\langle u_{n(k(\ell))}, x\right\rangle > \|u_{n(k(\ell))}\|^2 + \|x\|^2 - \epsilon > \epsilon$. Consequently,

$$\sum_{n\in\mathbb{N}} |\langle x, u_n\rangle|^2 \geq \sum_{\ell=N}^{\infty} \left|\left\langle x, u_{n(k(\ell))}\right\rangle\right|^2 \geq \sum_{l=N}^{\infty} \frac{\epsilon^2}{4} = \infty.$$

This contradicts the frame condition of $\boldsymbol{u}$. □

If $\boldsymbol{u}$ is not norm bounded from below, $(\kappa_\lambda)_{\lambda\in\Lambda}$ can have one or more accumulation points as the following elementary example shows.

Example 9. *Let* $\mathbb{X} = \mathbb{Y} = \ell^2(\mathbb{N})$ *and consider the diagonal multiplication operator* $\mathbf{K}\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})\colon (x_i)_{i\in\mathbb{N}} \mapsto \left(x_i/\sqrt{i+1}\right)_{i\in\mathbb{N}}$. *Clearly* $\mathbf{K}$ *is self-adjoint and compact with SVD given by* $((e_i)_{i\in\mathbb{N}}, (e_i)_{i\in\mathbb{N}}, (1/\sqrt{i+1})_{i\in\mathbb{N}})$ *where* $(e_i)_{i\in\mathbb{N}}$ *denotes the standard basis of* $\ell^2(\mathbb{N})$. *Define*

$$\begin{aligned}
\boldsymbol{u} &:= \left(e_0, e_0 \mid \frac{e_1}{\sqrt{2}}, \frac{e_1}{\sqrt{2}}, \frac{e_1}{\sqrt{2}} \mid \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}} \mid \dots\right) \\
\boldsymbol{v} &:= \left(e_0, e_0 \mid e_1, \frac{e_1}{\sqrt{2}}, \frac{e_1}{\sqrt{2}} \mid, e_2, \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}} \mid \dots\right) \\
\boldsymbol{\kappa} &:= \left(1, 1 \mid 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \mid 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \mid \dots\right).
\end{aligned}$$

For $x \in \ker(\mathbf{K})^\perp = \mathbb{X}$ *and* $y \in \overline{\operatorname{ran}(\mathbf{K})} = \mathbb{Y}$ *we have*

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle x, u_\lambda\rangle|^2 &= \sum_{n\in\mathbb{N}} (n+2) \left|\left\langle x, \frac{e_n}{\sqrt{n+1}}\right\rangle\right|^2 \\
&= \sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 + \sum_{n\in\mathbb{N}} \frac{1}{n+1} |\langle x, e_n\rangle|^2 \\
\sum_{\lambda\in\Lambda} |\langle y, v_\lambda\rangle|^2 &= \sum_{n\in\mathbb{N}} |\langle y, e_n\rangle|^2 + \sum_{n\in\mathbb{N}} (n+1) \left|\left\langle y, \frac{e_n}{\sqrt{n+1}}\right\rangle\right|^2 \\
&= \|y\|^2 + \|y\|^2 = 2\|y\|^2 .
\end{aligned}$$

Hence $\boldsymbol{u}$ *is a frame with frame bounds* $A = 1$ *and* $B = 2$ *and* $\boldsymbol{v}$ *is a frame with bounds* $A = B = 2$. *Moreover, the quasi-singular value relation* $\mathbf{K}^* v_\lambda = \kappa_\lambda u_\lambda$ *holds. Therefore* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ *is a DFD for the compact operator* $\mathbf{K}$. *However, the sequence* $\boldsymbol{\kappa}$ *has accumulation points* 0 *and* 1.

Note that we can easily modify example 9 such that ∞ is an accumulation point of $\boldsymbol{\kappa}$. To see this consider $\mathbf{K}$ and $\boldsymbol{v}$ from the example above and change $\boldsymbol{u}$ and $\boldsymbol{\kappa}$ to

$$\begin{aligned}
\boldsymbol{u} &= \left(e_0, e_0 \mid \frac{e_1}{2}, \frac{e_1}{\sqrt{2}}, \frac{e_1}{\sqrt{2}} \mid \frac{e_2}{3}, \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}}, \frac{e_2}{\sqrt{3}} \mid \dots\right) \\
\boldsymbol{\kappa} &= \left(1, 1 \mid \sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \mid \sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \mid \dots\right).
\end{aligned}$$

Then $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ is still a valid DFD of $\mathbf{K}$ where $\boldsymbol{u}$ has frame bounds $A = 1$ and $B = 2$, and $\boldsymbol{\kappa}$ has accumulation points 0 and ∞.

3 Regularization by filtered DFD

Throughout this section, let $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ be a DFD of the operator $\mathbf{K}$ and $\bar{\boldsymbol{u}}$ a dual frame of $\boldsymbol{u}$. For typical inverse problems, the Moore Penrose inverse is unbounded and therefore has to be regularized. In this section we develop a regularization concept by filtered diagonal frame decompositions.

3.1 Filtered DFD

Definition 10 (Regularizing filter)**.** We call a family $(f_\alpha)_{\alpha>0}$ of piecewise continuous functions $f_\alpha\colon (0,\infty) \to \mathbb{R}$ a regularizing filter if,

(F1) $\forall \alpha > 0 : \|f_\alpha\|_\infty < \infty$

(F2) $\exists C > 0 : \sup\{|z f_\alpha(z)| \mid \alpha > 0 \wedge z \geq 0\} \leq C$.

(F3) $\forall z \in (0,\infty) : \lim_{\alpha\to 0} f_\alpha(z) = 1/z$.

Using a regularizing filter we define the following central concept of this paper.

Definition 11 (Filtered diagonal frame decomposition)**.** Let $(f_\alpha)_{\alpha>0}$ be a regularizing filter and define

$$\forall \alpha > 0\colon \quad \mathbf{F}_\alpha\colon \mathbb{Y} \to \mathbb{X}\colon y \mapsto \sum_{\lambda\in\Lambda} f_\alpha(\kappa_\lambda) \langle y, v_\lambda\rangle \, \bar{u}_\lambda \, . \tag{15}$$

We call the family $(\mathbf{F}_\alpha)_{\alpha>0}$ the filtered diagonal frame decomposition (filtered DFD) according to $(f_\alpha)_{\alpha>0}$ based on the DFD $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ and the dual frame $\bar{\boldsymbol{u}}$.

In this paper we show that filtered DFD yields a regularization method. To that we first recall the definition of regularization method.

Definition 12 (Regularization method)**.** Let $(\mathbf{R}_\alpha)_{\alpha>0}$ a family of continuous operators $\mathbf{R}_\alpha\colon \mathbb{Y} \to \mathbb{X}$, $y \in \operatorname{dom}(\mathbf{K}^\dagger)$ and $\alpha^*\colon (0,\infty)\times\mathbb{Y} \to (0,\infty)$. Then the pair $((\mathbf{R}_\alpha)_{\alpha>0}, \alpha^*)$ is a regularization method for the solution of $\mathbf{K}\, x = y$, if

$$\begin{aligned}
&\lim_{\delta\to 0} \sup\{\alpha^*(\delta, y^\delta) \mid y^\delta \in \mathbb{Y} \wedge \|y^\delta - y\| \leq \delta\} = 0 \\
&\lim_{\delta\to 0} \sup\{\| \mathbf{K}^\dagger\, y - \mathbf{R}_{\alpha^*(\delta,y^\delta)}\, y^\delta\| \mid y^\delta \in \mathbb{Y} \wedge \|y^\delta - y\| \leq \delta\} = 0 \, .
\end{aligned}$$

In this case we call α^* an admissible parameter choice. If for any $y \in \operatorname{dom}(\mathbf{K}^\dagger)$ there exists an admissible parameter choice, then we call $(\mathbf{R}_\alpha)_{\alpha>0}$ a regularization of the Moore Penrose inverse $\mathbf{K}^\dagger$.

Given an SVD $(u_n, v_n, \sigma_n)_{n\in\mathbb{N}}$ of $\mathbf{K}$ and a regularizing filter $(f_\alpha)_{\alpha>0}$, it is well known that the family

$$\sum_{n\in\mathbb{N}} g_\alpha(\sigma_n^2) \langle \mathbf{K}^*\, y, u_n\rangle \, u_n = \sum_{n\in\mathbb{N}} f_\alpha(\sigma_n) \langle y, v_n\rangle \, u_n = \mathbf{F}_\alpha(y)$$

with $f_\alpha(\sigma_n) = \sigma_n g_\alpha(\sigma_n^2)$ defines a regularization method [7] together with convergence rates. Two prominent examples of filter-based regularization methods are classical Tikhonov regularization and truncated SVD. In truncated SVD, the regularizing filter is given by

$$f_\alpha(\sigma) = \begin{cases} 0 & \text{if } \sigma^2 < \alpha \\ 1/\sigma & \text{if } \sigma^2 \geq \alpha \, . \end{cases} \tag{16}$$

In Tikhonov regularization, the regularizing filter is given by $f_\alpha(\sigma) = \sigma/(\sigma^2 + \alpha)$, which yields in the explicit expression $\mathbf{F}_\alpha = (\mathbf{K}^*\, \mathbf{K} + \alpha \operatorname{Id}_{\mathbb{X}})^{-1}\, \mathbf{K}^*$.

In this paper we generalize such results by allowing a DFD instead of the SVD. To that end we use the following well known result.

Lemma 13 (Characterization of linear regularizations)**.** *Let* $(\mathbf{R}_\alpha)_{\alpha>0}$ *be a family of linear bounded operators which pointwise converge to* $\mathbf{K}^\dagger$ *on* $\operatorname{dom}(\mathbf{K}^\dagger)$ *and let* $y \in \operatorname{dom}(\mathbf{K}^\dagger)$*. If the parameter choice* $\alpha^\star \colon (0,\infty) \to (0,\infty)$ *satsfies* $\lim_{\delta\to 0} \alpha^\star(\delta) = \lim_{\delta\to 0} \delta \| \mathbf{R}_{\alpha^\star(\delta)} \| = 0$*, then the pair* $((\mathbf{R}_\alpha)_{\alpha>0}, \alpha^\star)$ *is a regularization method for* $\mathbf{K}\, x = y$*.*

Proof. See, for example, [7]. □

3.2 Convergence analysis for filtered DFD

Let $(f_\alpha)_{\alpha>0}$ be a regularizing filter and $(\mathbf{F}_\alpha)_{\alpha>0}$ be the filtered DFD defined by (15).

Proposition 14 (Existence and stability of filtered DFD)**.** *For any* $\alpha > 0$ *the operator* $\mathbf{F}_\alpha$ *is well defined and bounded. Moreover,* $\|\mathbf{F}_\alpha\| \le \|f_\alpha\|_\infty \sqrt{B_{\bar{\mathbf{u}}} B_{\boldsymbol{v}}}$*, where* $B_{\bar{\mathbf{u}}}$ *and* $B_{\boldsymbol{v}}$ *are the upper frame bounds of* $\bar{\mathbf{u}}$ *and* $\boldsymbol{v}$*, respectively.*

Proof. First we show that $\mathbf{F}_\alpha$ is continuous for all $\alpha > 0$. Fix $\alpha > 0$. Since $\|f_\alpha\|_\infty < \infty$ we have $(f_\alpha(\kappa_\lambda) \langle y, v_\lambda \rangle)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ and we can find a bound for $\mathbf{F}_\alpha$. To that end we estimate

$$
\begin{aligned}
\|\mathbf{F}_\alpha\, y\|^2 &= \left\| \sum_{\lambda\in\Lambda} f_\alpha(\kappa_\lambda) \langle y, v_\lambda \rangle \bar{u}_\lambda \right\|^2 \\
&\le \|T^*_{\bar{\mathbf{u}}}\|^2 \sum_{\lambda\in\Lambda} |f_\alpha(\kappa_\lambda) \langle y, v_\lambda \rangle|^2 \\
&= \|T^*_{\bar{\mathbf{u}}}\|^2 \sum_{\lambda\in\Lambda} |f_\alpha(\kappa_\lambda)|^2 |\langle y, v_\lambda \rangle|^2 \\
&\le \|T^*_{\bar{\mathbf{u}}}\|^2 \|f_\alpha\|_\infty^2 \sum_{\lambda\in\Lambda} |\langle y, v_\lambda \rangle|^2 \\
&\le \|f_\alpha\|_\infty^2 B_{\bar{\mathbf{u}}} B_{\boldsymbol{v}} \|y\|^2 .
\end{aligned}
$$

This shows that $\mathbf{F}_\alpha\, y$ is well defined, and that $\|\mathbf{F}_\alpha\| \le \|f_\alpha\| \sqrt{B_{\bar{\mathbf{u}}} B_{\boldsymbol{v}}}$. □

Proposition 15 (Pointwise convergence of filtered DFD)**.** *For all* $y \in \operatorname{dom}(\mathbf{K}^\dagger)$ *we have* $\lim_{\alpha\to 0} \mathbf{F}_\alpha\, y = \mathbf{K}^\dagger(y)$*.*

Proof. Let $y = \hat{y} + y_0 \in \operatorname{ran}(\mathbf{K}) \oplus \operatorname{ran}(\mathbf{K})^\perp$ and $x^\dagger := \mathbf{K}^\dagger(y)$. Then it holds that $\mathbf{K}\, x^\dagger =$

$\mathbf{P}_{\overline{\operatorname{ran}(\mathbf{K})}}(y) = \hat{y}$ and $x^\dagger \in \ker(\mathbf{K})^\perp$ and therefore

$$\begin{aligned}
\left\|x^\dagger - \mathbf{F}_\alpha\, y\right\|^2 &= \left\| x^\dagger - \sum_{\lambda\in\Lambda} f_\alpha(\kappa_\lambda)\, \langle y, v_\lambda \rangle\, \bar{u}_\lambda \right\|^2 \\
&= \left\| x^\dagger - \sum_{\lambda\in\Lambda} f_\alpha(\kappa_\lambda)\, \langle \hat{y}, v_\lambda \rangle\, \bar{u}_\lambda \right\|^2 \\
&= \left\| x^\dagger - \sum_{\lambda\in\Lambda} f_\alpha(\kappa_\lambda)\, \left\langle \mathbf{K}\, x^\dagger, v_\lambda \right\rangle\, \bar{u}_\lambda \right\|^2 \\
&= \left\| \sum_{\lambda\in\Lambda} \left\langle x^\dagger, u_\lambda \right\rangle \bar{u}_\lambda - \sum_{\lambda\in\Lambda} \kappa_\lambda f_\alpha(\kappa_\lambda) \left\langle x^\dagger, u_\lambda \right\rangle \bar{u}_\lambda \right\|^2 \\
&= \left\| \sum_{\lambda\in\Lambda} (1 - \kappa_\lambda f_\alpha(\kappa_\lambda)) \left\langle x^\dagger, u_\lambda \right\rangle \bar{u}_\lambda \right\|^2 \\
&\leq B_{\bar{\boldsymbol{u}}} \sup_{\lambda\in\Lambda} (1 - \kappa_\lambda f_\alpha(\kappa_\lambda))^2 \sum_{\lambda\in\Lambda} \left|\left\langle x^\dagger, u_\lambda \right\rangle\right|^2 \\
&\leq \sup_{\lambda\in\Lambda} (1 - \kappa_\lambda f_\alpha(\kappa_\lambda))^2 B_{\bar{\boldsymbol{u}}} B_{\boldsymbol{u}} \left\|x^\dagger\right\|^2 .
\end{aligned}$$

Moreover, we have $\lim_{\alpha\to 0}(1 - \kappa_\lambda f_\alpha(\kappa_\lambda)) = 0$ and $\sup_{\lambda\in\Lambda}(1 - \kappa_\lambda f_\alpha(\kappa_\lambda))^2 < \infty$. Therefore, the dominated convergence theorem yields $\|x^\dagger - \mathbf{F}_\alpha\, y\|^2 \to 0$ for $\alpha \to 0$. □

By collecting the above results we obtain the following main convergence result for filtered DFD.

Theorem 16 (Convergence of filtered DFD)**.** *Let $(f_\alpha)_{\alpha>0}$ be a regularizing filter, $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ be a DFD of the compact operator $\mathbf{K}$ and $\bar{\boldsymbol{u}}$ a dual frame of $\boldsymbol{u}$. Moreover, let $\alpha^*\colon (0,\infty) \to (0,\infty)$. Then $((\mathbf{F}_\alpha)_{\alpha>0}, \alpha^*)$ is a regularization method for $\mathbf{K}\, x = y$ provided that parameter choice satisfies*

$$0 = \lim_{\delta\to 0} \alpha^*(\delta) = \lim_{\delta\to 0} \delta \|f_{\alpha^*(\delta)}\|_\infty .$$

Proof. According to Propositions 14 and 15, $(\mathbf{F}_\alpha)_{\alpha>0}$ is a family of bounded linear operators that converges pointwis to $\mathbf{K}^\dagger$ on $\operatorname{dom}(\mathbf{K})$. According to Lemma 13 the pair $((\mathbf{F}_\alpha)_{\alpha>0}, \alpha^*)$ is a regularization method if $\alpha^\star(\delta)$ and $\delta\| \mathbf{F}_{\alpha^*(\delta)} \|$ converge to zero. The estimate $\|\mathbf{F}_\alpha\| \leq \|f_\alpha\|_\infty \sqrt{B_{\bar{\boldsymbol{u}}} B_{\boldsymbol{v}}}$ of Proposition 14 then yields the claim. □

3.3 Convergence rates for filtered DFD

Next we derive convergence rates which give quantitative estimates on the reconstruction error $\left\|x^\dagger - x_\alpha^\delta\right\|$.

Theorem 17 (Convergence rates for filtered DFD)**.** *Let $(f_\alpha)_{\alpha>0}$ be a regularizing filter, $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ be a DFD of the compact operator $\mathbf{K}$ and $\bar{\boldsymbol{u}}$ a dual frame of $\boldsymbol{u}$ and $(\mathbf{F}_\alpha)_{\alpha>0}$ be the filtered DFD defined by* (15)*. For given numbers $\rho, \mu > 0$ suppose*

(R1) $\|f_\alpha\|_\infty = \mathcal{O}(\alpha^{-1})$ *as* $\alpha \to 0$,

(R2) $\forall \alpha > 0\colon \sup\{\kappa^\mu \left|1 - \kappa f_\alpha(\kappa)\right| \mid \kappa \in (0,\infty)\} \le \tilde{C}\alpha^\mu$,

(R3) $\alpha = \alpha^*(\delta, y^\delta) \asymp (\delta/\rho)^{1/(\mu+1)}$.

Suppose $x^\dagger \in \mathbb{X}$ satisfying the following source type condition

$$\exists \boldsymbol{\omega} \in \ell^2(\Lambda)\colon \|\boldsymbol{\omega}\|_2 \le \rho \text{ and } \forall \lambda \in \Lambda\colon \left\langle x^\dagger, u_\lambda \right\rangle = \kappa_\lambda^\mu \omega_\lambda \,. \tag{17}$$

Then, for some constant $c = c(\rho, \mu)$ and all $y^\delta \in \mathbb{Y}$ with $\left\|y^\delta - \mathbf{K}\, x^\dagger\right\| \le \delta$ with sufficiently small δ, the following convergence rate result holds:

$$\left\|x^\dagger - \mathbf{F}_{\alpha^*}(y^\delta)\right\| \le c(\rho,\mu)\, \delta^{\frac{\mu}{\mu+1}} \,.$$

Proof. Let $x^\dagger$, $\boldsymbol{\omega}$, y^δ satisfy $\left\langle x^\dagger, u_\lambda \right\rangle = \kappa_\lambda^\mu \omega_\lambda$, $\|\boldsymbol{\omega}\|_{\ell^2} \le \rho$, $\left\|y^\delta - \mathbf{K}\, x^\dagger\right\| \le \delta$. Then

$$\begin{aligned}
\left\|\mathbf{F}_\alpha(y^\delta) - x^\dagger\right\| &\le \left\|\mathbf{F}_\alpha(y^\delta - \mathbf{K}\, x^\dagger)\right\| + \left\|\mathbf{F}_\alpha(\mathbf{K}\, x^\dagger) - x^\dagger\right\| \\
&\le \|\mathbf{F}_\alpha\|\delta + \left\|\sum_{\lambda\in\Lambda} (1 - \kappa_\lambda f_\alpha(\kappa_\lambda)) \left\langle x^\dagger, u_\lambda \right\rangle \bar{u}_\lambda\right\| \\
&\le \sqrt{B_{\bar{\boldsymbol{u}}} B_{\boldsymbol{v}}}\|f_\alpha\|\delta + \left(B_{\bar{\boldsymbol{u}}} \sum_{\lambda\in\Lambda} \left|1 - \kappa_\lambda f_\alpha(\kappa_\lambda)\right|^2 \left|\left\langle x^\dagger, u_\lambda \right\rangle\right|^2\right)^{\frac{1}{2}} \\
&\le c_1 \alpha^{-1}\delta + \left(B_{\bar{\boldsymbol{u}}} \sum_{\lambda\in\Lambda} \left|1 - \kappa_\lambda f_\alpha(\kappa_\lambda)\right|^2 \left|\kappa_\lambda^\mu \omega_\lambda\right|^2\right)^{\frac{1}{2}} \\
&\le c_1 \alpha^{-1}\delta + \sqrt{B_{\bar{\boldsymbol{u}}}}\tilde{C}\alpha^\mu \rho.
\end{aligned}$$

Now choose $\alpha = \alpha^*(\delta, y^\delta) \asymp (\delta/\rho)^{\frac{1}{\mu+1}}$. Then the above estimate implies

$$\left\|\mathbf{F}_{\alpha^*(\delta,y^\delta)}(y^\delta) - x^\dagger\right\| \le c_2 \left(\delta^{1-\frac{1}{\mu+1}} \rho^{\frac{1}{\mu+1}} + \delta^{\frac{\mu}{\mu+1}} \rho^{1-\frac{\mu}{\mu+1}}\right) = \mathcal{O}\left(\delta^{\frac{\mu}{\mu+1}}\right) \,,$$

and completes the proof. □

In the following we prove that the convergence rates obtained in Theorem 17 are order optimal for the source set defined by (17) in the special case that the frame $\boldsymbol{u}$ admits an biorthogonal sequence $\bar{\boldsymbol{u}} = (u_\lambda)_{\lambda\in\Lambda}$ such that

$$\forall \lambda, \mu \in \Lambda\colon \quad \langle u_\lambda, \bar{u}_\nu \rangle = \delta_{\lambda,\nu} \,. \tag{18}$$

Note that the requirement that $\boldsymbol{u}$ has a biorthogonal sequence is equivalent to $\boldsymbol{u}$ being a Riesz-basis of $\ker(\mathbf{K})^\perp$.

To do this we define

$$U_{\mu,\delta} := \{x \mid \langle x, u_\lambda \rangle = \kappa_\lambda^\mu w_\lambda \wedge \sum_{\lambda\in\Lambda} |w_\lambda|^2 = \rho^2\} \tag{19}$$

and for any set $\mathcal{M} \subseteq \mathbb{X}$ define $\epsilon(\mathcal{M}, \delta) := \sup\{\|x\| \mid x \in \mathcal{M} \wedge \|\mathbf{K}\, x\| \le \delta\}$. We have that $\epsilon(\mathcal{M}, \delta)$ is a lower bound for the worst case reconstruction error

$$E(\mathcal{M}, \delta, \mathbf{R}) := \sup\{\|\,\mathbf{R}\, y - x\| \mid x \in \mathcal{M} \wedge y^\delta \in \mathbb{Y} \wedge \left\|\mathbf{K}\, x - y^\delta\right\| \le \delta\} \,, \tag{20}$$

for an arbitrary reconstruction method $\mathbf{R}\colon \mathbb{Y} \to \mathbb{X}$ with $\mathbf{R}(0) = 0$ applied to noisy data [7]. A family $(\mathbf{R}^\delta)_{\delta>0}$ of reconstruction methods is called order optimal on $\mathcal{M}$, if we have $E(\mathcal{M}, \delta, \mathbf{R}^\delta) \leq c\,\epsilon(\mathcal{M}, \delta)$ for sufficiently small δ and some constant $c > 0$. To show that the convergence rate of Theorem 17 is order optimal therefore amounts to bound $\epsilon(U_{\mu,\rho}, \delta)$.

Theorem 18. *Let $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa})$ be a DFD of the compact operator $\mathbf{K}$ such that $\boldsymbol{u}$ has a biorthogonal sequence $\bar{\boldsymbol{u}}$. Then for the source sets $U_{\mu,\rho}$ defined by (19) we have*

$$\epsilon(U_{\mu,\rho}, \delta) \leq \sqrt{\tfrac{B_{\boldsymbol{v}}}{A_{\boldsymbol{u}}}}\, \delta^{\frac{\mu}{\mu+1}} \rho^{\frac{1}{\mu+1}}.$$

In particular, under the assumptions of Theorem 17, the family $(\mathbf{F}_{\alpha^*(\delta,\cdot)})_{\delta>0}$ is an order optimal reconstruction method for the source set $U_{\mu,\rho}$.

Proof. Let $x_\nu := \rho\kappa_\nu^\mu \bar{u}_\nu$ such that

$$\langle x_\nu, u_\lambda\rangle = \kappa_\lambda^\mu w_\lambda, \quad w_\lambda = \begin{cases} \rho, & \text{if } \lambda = \nu \\ 0, & \text{else}\,. \end{cases}$$

By definition we have $\|w\|_2 = \rho$ and $x_\nu \in U_{\mu,\rho}$. If we consider the decreasing null-sequence of noise levels $\delta_\nu = \rho\kappa_\nu^{\mu+1}/\sqrt{A_{\boldsymbol{v}}}$ we get that:

$$\|x_\nu\|^2 \geq \tfrac{1}{B_{\boldsymbol{u}}} \sum_{\lambda\in\Lambda} |\langle u_\lambda, x_\nu\rangle|^2 = \tfrac{1}{B_{\boldsymbol{u}}} \kappa_\nu^{2\mu} \rho^2 = A_{\boldsymbol{v}}^{\mu/(\mu+1)} \tfrac{1}{B_{\boldsymbol{u}}} \left(\delta_\nu^{\mu/(\mu+1)} \rho^{1/(\mu+1)}\right)^2$$

and

$$\begin{aligned} \|\mathbf{K}\,x_\nu\|^2 &\leq \tfrac{1}{A_{\boldsymbol{v}}} \sum_{\lambda\in\Lambda} |\langle v_\lambda, \mathbf{K}\,x_\nu\rangle|^2 \\ &= \tfrac{1}{A_{\boldsymbol{v}}} \sum_{\lambda\in\Lambda} \kappa_\lambda^2 |\langle u_\lambda, x_\nu\rangle|^2 \\ &= \tfrac{1}{A_{\boldsymbol{v}}} \kappa_\nu^{2(\mu+1)} \rho^2 = \delta_\nu^2\,. \end{aligned}$$

Thus, for δ_ν and x_ν we have $\|\mathbf{K}\,x_\nu\| \leq \delta_\nu$ and hence

$$\epsilon(U_{\mu,\rho}, \delta) \geq \|x_\nu\| \geq \sqrt{\tfrac{A_{\boldsymbol{v}}}{B_{\boldsymbol{u}}}}\, \delta_\nu^{\mu/(\mu+1)} \rho^{1/(\mu+1)}.$$

□

4 Conclusion and outlook

In this work we analyzed the concept of diagonal frame decomposition (DFD) for the solution of linear inverse problems. A DFD for the operator $\mathbf{K}$ yields the explicit formula $\mathbf{K}^\dagger = \mathbf{T}_{\bar{\boldsymbol{u}}}^* \mathbf{M}_{1/\kappa} \mathbf{T}_{\boldsymbol{v}}$ for the pseudoinverse. In the ill-posed case, the pseudoinverse $\mathbf{K}^\dagger$ is unbounded as well as is the sequence $1/\kappa$. We showed that replacing the $1/\kappa_\lambda$ by a regularized filter (Definition 10) applied to the quasi-singular values κ_λ results in a regularization method (Theorem 16). As another main result we derived convergence rates for filtered DFD in Theorem 17. By noting that the DFD reduced to the SVD in the case of orthogonal basis, we see that our results extend well-known convergence and convergence rates results of filter based SVD regularization [7, 12] to the DFD case.

One advantage of filtered DFD regularization over variational regularization methods is their explicit form. Compared to SVD based regularization, benefits are that a DFD may be available even when no SVD is known or has to be computed numerically. Moreover, the associated analysis and synthesis operations can often be implemented for the DFD. The use of the DFD is of practical relevance as frames such as wavelets or curvelet have better approximation capabilities for typical images to be reconstructed [1] than singular systems. In order to fully exploit such properties a main aspect of future research is to extend the presented convergence analysis to non-linear filters. As a first step in this direction see the work [9] where a convergence analysis is presented using soft-thresholding defining a non-linear filtered DFD.

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