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Gerhard Zangerl

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Technikerstraße 13 - 6020 Innsbruck - Austria
Tel.: +43 512 507 53803 Fax: +43 512 507 53898
https://applied-math.uibk.ac.at

On the Configuration Space of planar closed Kinematic Chains

Gerhard Zangerl *

Abstract

A planar kinematic chain consists of n links connected by joints. In this work we investigate the space of configurations, described in terms of joint angles, that guarantee that the kinematic chain is closed. We give explicit formulas expressing the joint angles that guarantee closedness by a new set of parameters. Moreover, it turns out that this parameters are contained in a domain that possesses a simple structure. We expect that the new insight can be applied for several issues such as motion planning for closed kinematic chains or singularity analysis of their configuration spaces. In order to demonstrate practicality of the new method we present numerical examples.

Keywords. configuration space, path planning, closed kinematic chain

Contents

*Department of Mathematics, University of Innsbruck, Technikerstrae 13, A-6020 Innsbruck, Austria, E-mail: Gerhard.Zangerl@uibk.ac.at

1 Introduction

In this work we investigate the configuration space of closed kinematic chain (CKC) with n links connected by revolute joints in terms of its joint angles. In many fields like robotics computational biology or protein kinematics it is of immense interest to understand the configuration space of a CKC. For instance, in robotics the problem to connect a start, α_s, and goal configuration, α_g naturally appears and thus requires knowledge of the configuration space, which is typically a manifold or variety in the ambient space formed by the robots joint variables. The configuration space is even more complicated if additional constraints like obstacle, link-link avoidance, or limited joint angles are included. Two main strategies, probabilistic and geometric approaches, to investigate configuration spaces have been developed so far.

Probabilistic methods have been successfully applied for constrained motion planning. They are especially important in practical situations with high dimensions that include complex constraints such as obstacle avoiding. Typically this methods are based on the generation of random configurations in ambient joint space followed by a check up if they approximately satisfy the desired constrains. Repeating this procedure results in a discrete version of the configuration space that is very useful in applications. Probabilistic methods have been applied in different situations, which can be found in Cortes & Simeon (2004), S. La Valle & Kavraki (1999), Valle (2006), Yajia et al. (2013), Chirikjian (2000), Jaillet & Porta (2013), Yakey et al. (2001) Suh et al. (2011).

Besides the approaches using randomness other works focused on questions about the geometry and topology of the configuration spaces of kinematic chains. Insights about the global geometry of configuration spaces is very important in applications. Early discoveries have been made by Kapovich & J.Millson (1995), Hausmann & Knutson (1998). In their fundamental work Kapovitch and Milgram established important results about the geometry, which led to novel path planning algorithms. For instance in Milgram et al. (2004), Trinkle & Milgram (2002) it is used that the configuration space of a CKC consists of two connected components when it possess three long links. An application of this result is that path planning can be done easily for this special kind of CKC's. Also for the more difficult case, when CKC's do nit have three long links algorithms were derived in Milgram et al. (2004), Trinkle & Milgram (2002). They also developed path planners in the case of p point obstacles in the plane G. F. Liu & Milgram (2005). Another geometric approach was recently recognized by Han, Rudolph and Blumenthal. They discovered that it is very beneficial to describe the configuration space of CKC by different parameters than the joint angles, see Han et al. (2008b,a), Han & Rudolph (2006). Their idea is to use the length of diagonals from the positions of revolute joints to the origin O as depicted on the right side of Figure 2.1. It turns out that for a CKC the length of these diagonals can be computed as solution of a system of linear inequalities, which means that all feasible diagonal lengths can be described by a convex polyhedron that can be handled by methods of linear programming Luenberger & Yinyu (2015). Given feasible diagonal lengths, several configurations of the CKC can be constructed, since each link of the chain can be flipped over a diagonal. Thus in Han et al. (2008b,a) any configuration can be obtained from a set of diagonals and a vector that represents the choices of flipping, which shows that the configuration space is formed by several copies of the polyhedron given by the system of inequalities. This practically convex structure is very useful for motion planning. In Han & Rudolph (2006), Han et al. (2008*a*) paths between CKC with 1000 links are computed very efficiently.

Contribution of this work: We develop a new method that to explicitly computes configurations of a CKC with n links, which are described by its joint angles. Compared to other methods it does not require linear programming to solve a system of linear inequalities like in Han et al. (2008*b*,*a*) nor does it rely on probabilistic principles. More precisely, it turns out that a configuration can be computed from new parameters contained in a very simple domain, namely a $n-3$ dimensional cube. The developed method can be used to easily sample configuration space of a CKC and thus is expected to be useful in practical applications.

Outline of this text: In section 2 we give a mathematical description of a CKC and its configuration space. Then the basic algorithm that explicitly describes how configurations of a CKC can be computed is developed in section 2. In section 3.1 we describe the set of new parameters and show how they can be used to compute a vector of joint angles of a CKC. Finally, we give a numerical examples that show validity of the developed method.

2 Configuration space

To describe the configuration space of a CKC with link lengths $a_1, \dots, a_n$ we introduce Cartesian coordinates in two dimensional Euclidean space. Moreover we place one of the links of the CKC so that it is supported by the positive x-axis and so that one of its ends coincides with the origin. Without loss of generality we can assume that the link a_n of the chain is fixed in the described manner, see Figure 1. In the following, we identify an angle α with its corresponding point on S^1. Further, for $1 \leq k \leq n$ and a vector of angles $\alpha^k := (\alpha_1, \dots, \alpha_k) \in \left(S^1\right)^k$ we denote by

$$f_{a,k} \colon \left(S^1\right)^k \to \mathbb{R}^2, \quad f_{a,k}(\alpha^k) = \sum_{j=1}^{k} a_j \begin{pmatrix} \cos(\alpha_j) \\ \sin(\alpha_j) \end{pmatrix}. \tag{1}$$

the $k-$th endpoint map of a kinematic chain, where $a = (a_1, \dots, a_n)$ is the vector of link lengths. We will call α^{n-1} a configuration of the CKC with link lengths $a_1, \dots, a_n$ if it satisfies the closure condition, which means that it is contained in the set

$$\mathrm{C}_a = \left\{ \alpha^{n-1} \in \left(S^1\right)^{n-1} : f_{a,n-1}\left(\alpha^{n-1}\right) = \begin{pmatrix} a_n \\ 0 \end{pmatrix} \right\} = f^{-1}_{a,n-1}\left(a_n, 0\right). \tag{2}$$

If no restrictions on the endpoint map are imposed α^{n-1} will just be called a configuration of the kinematic chain (KC) with $n-1$ links.

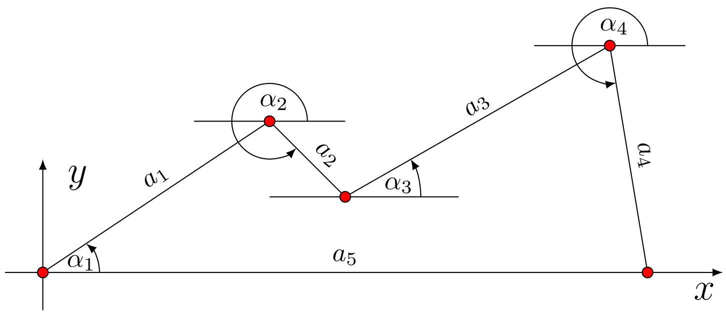

Figure 1: A CKC with $n = 5$ five links. The link a_5 is supported on the positive x-axis and one of its ends coincides with the origin

Furthermore, the analysis in this work uses the simple observation that it is sufficient to understand the space

$$\mathcal{CC}_a = \left\{ \beta^{n-1} \in \left(S^1\right)^{n-1} : \|f_{a,n-1}\left(\beta^{n-1}\right)\|_2^2 = a_n^2 \right\}, \tag{3}$$

in order to describe C_a, where $\|\cdot\|_2$ denotes the euclidean norm. From the definition of $\mathcal{CC}_a$ it is clear that any configuration $\beta^{n-1} \in \mathcal{CC}_a$ satisfies that its endpoint

$$f_{a,n-1}\left(\beta^{n-1}\right) \in K_{a_n}$$

lies on the circle K_{a_n} that is centred on the origin and has radius a_n. We will say that β^{n-1} is closed up to a rotation and call it a *circular configuration* of a CKC. Clearly, any circular configuration β^{n-1} can be rotated by an angle λ,

$$\beta^{n-1} + \lambda := (\beta_1 + \lambda, \dots, \beta_{n-1} + \lambda)\,,$$

so that $\beta^{n-1} + \lambda \in C_a$. Thus, if we are able to give an efficient method to compute the set of solutions to the implicit equation

$$\| f_{a,n-1}\left(\beta^{n-1}\right) \|^2 = a_n^2, \tag{4}$$

we also obtain configurations in C_a by the following two step algorithm:

(i) Compute a circular configuration $\beta^{n-1} \in \mathcal{CC}_a$

(ii) Determine λ such that $\alpha^{n-1} = \beta^{n-1} + \lambda \in C_a$

Once a circular configuration is obtained step (ii) is a rather simple task. Therefore, in the following we will focus on the solution of step (i). This step is based on the fact that the trigonometric equation (4), which in its expanded form is given as

$$\sum_{i=1}^{n-1} a_i^2 + 2 \sum_{i<j}^{n-1} a_i a_j \cos(\beta_i - \beta_j) = a_n^2, \tag{5}$$

allows for some kind of backwards substitution, see section 2.2. By the preimage theorem we know that the set of all circular configurations of a CKC with n links satisfying (5) is a manifold of dimension $n-2$, whenever a_n^2 is a regular value of the map $g\left(\beta^{n-1}\right) := \| f_{a,n-1}\left(\beta^{n-1}\right) \|_2^2$. In all other cases the space $\mathcal{CC}_a$ may have singular points.

2.1 Mathematical tools and notations

Surprisingly, the trigonometric equation (5) can be rearranged into an equation of the same type but with one joint angle less appearing on its left hand side. The computations requires that a linear combination of sine and cosine functions can be written as

$$a \sin(x) + b \cos(x) = c \sin(x + \varphi(a,b))\,,$$

where $c = \sqrt{a^2 + b^2}$ and $\varphi(a,b) = \mathrm{atan2}(b,a)$ is the function described in Figure 2.1. It is important to introduce some abbreviations, which make it possible to write down the following results in a compact manner:

For a CKC with link lengths $a_1, a_2, \dots, a_n$, a configuration $\beta^{n-1} = (\beta_1, \dots, \beta_{n-1})$

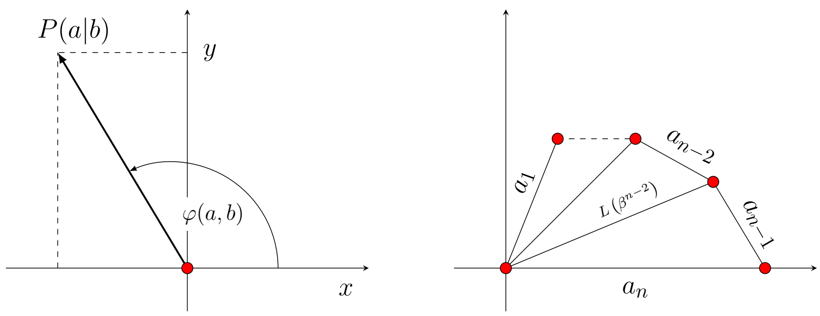

Figure 2: Left: The function atan2 gives the angle between the $x-$axis and the vector from the origin to $P(a|b)$. Right: A circular configuration with endpoint $(a_n, 0)$. The picture shows anchored diagonals of the CKC

and for $1 \leq k \leq n-1$ we introduce the abbreviations

$$\begin{aligned}
\beta^{n-k} &:= (\beta_1, \ldots, \beta_{n-k}) \\
S_{n-k} &:= \sum_{i=1}^{n-k} a_i^2, \quad X\left(\beta^{n-k}\right) := \sum_{i<j}^{n-k} a_i a_j \cos\left(\beta_i - \beta_j\right), \\
\Phi\left(\beta^{n-k}\right) &:= \operatorname{atan2}\left(\sum_{j=1}^{n-k} a_j \cos\left(\beta_j\right), \sum_{j=1}^{n-k} a_j \sin\left(\beta_j\right)\right), \\
L\left(\beta^{n-k}\right) &:= \sqrt{S_{n-k} + 2X(\beta^{n-k})},
\end{aligned}$$

where we assume $\beta^1 = \beta_1$ and $X\left(\beta^1\right) = 0$. For $3 \leq k \leq n-1$ let $C^k := (C_3, \ldots, C_k)$ be a vector with $k-2$ entries and let C_n be the constant given by

$$C_n := \frac{a_n^2 - S_{n-1}}{2}.$$

Furthermore, for $A \in \mathbb{R}$ let $\mathrm{E}_k(A)$ be the quadratic equation in the variable C corresponding to the value A given by

$$(A - C)^2 - a_{n-k}^2 \left(S_{n-k-1} + 2C\right) = 0$$

and let $A^{\pm}$ be its solutions.

2.2 Computation of circular configurations

We give a new method to obtain solutions to equation (5) in theorem 2.1. In this theorem solutions of equation (5) are obtained systematically by reducing the length of the CKC step by step. For $k \geq 1$ the solution method involves the choice of a real value C_{n-k} between the roots of the quadratic equation $\mathrm{E}_k(C_{n-k+1})$. This value C_{n-k} defines the next quadratic equation $\mathrm{E}_k(C_{n-k})$ and a new value C_{n-k+1} is choosen between its roots and so on. In order to guarantee that this procedure is well defined, we need that all quadratic equations connected in this manner have real solutions. Lemma 2.1 shows that the values C_{n-k} are indeed real.

Lemma 2.1. *Let C_{n-k+1} be a real number with $2C_{n-k+1} + S_{n-k} \geq 0$, which means that the roots $C^{\pm}_{n-k+1}$ of the quadratic equation $\mathrm{E}_k(C_{n-k+1})$, given by*

$$(C_{n-k+1} - C)^2 - a^2_{n-k}\left(S_{n-k-1} + 2C\right) = 0, \tag{6}$$

are real. Choosing a value C_{n-k} between its roots guarantees that equation $\mathrm{E}_{k+1}(C_{n-k})$ also has real solutions. Moreover, the solutions of $\mathrm{E}_1(C_n)$ are real.

Proof. For $k = 1$ we have that the roots $C^{\pm}_n$ of the equation $\mathrm{E}_1(C_n)$, given by

$$(C_n - C)^2 - a^2_{n-1}\left(S_{n-2} + 2C\right) = 0,$$

are real, since they are computed to be

$$\begin{aligned} C^{\pm}_n &= C_n + a^2_{n-1} \pm a_{n-1}\sqrt{2C_n + S_{n-1}} \\ &= \frac{a^2_n + a^2_{n-1}}{2} - \frac{S_{n-2}}{2} \pm a_{n-1}\sqrt{a^2_n - S_{n-1} + S_{n-1}} \\ &= \frac{a^2_n + a^2_{n-1}}{2} - \frac{S_{n-2}}{2} \pm a_{n-1}a_n. \end{aligned}$$

By our assumption the roots of $\mathrm{E}_k\left(C_{n-k+1}\right)$ are real. We have to prove that the roots of equation $\mathrm{E}_{k+1}\left(C_{n-k}\right)$ corresponding to C_{n-k} are still real, whenever $C^{-}_{n-k+1} \leq C_{n-k} \leq C^{+}_{n-k+1}$. This means that the roots of

$$(C_{n-k} - C)^2 - a^2_{n-k-1}\left(S_{n-k-2} + 2C\right) = 0$$

are real, which is the case if $2C_{n-k} + S_{n-k-1} \geq 0$. By our assumption C_{n-k} is real and clearly $C_{n-k} \geq C^{-}_{n-k+1}$. Thus

$$2C_{n-k} + S_{n-k-1} \geq 2C^{-}_{n-k+1} + S_{n-k-1}.$$

The Lemma follows, if we can show that the right side of the last inequality is greater than zero. We plug in the explicit expression

$$C^{-}_{n-k+1} = C_{n-k+1} + a^2_{n-k} - a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}}$$

into the latter inequality and obtain

$$\begin{aligned} 2\left(C_{n-k+1} + a^2_{n-k} - a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}}\right) + S_{n-k-1} \geq 0 \Leftrightarrow \\ 2C_{n-k+1} + \underbrace{\left(S_{n-k-1} + a^2_{n-k}\right)}_{=S_{n-k}} + a^2_{n-k} - 2a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}} \geq 0. \end{aligned}$$

By assumption $D := 2C_{n-k+1} + S_{n-k} \geq 0$ and thus the latter inequality is satisfied since

$$D + a^2_{n-k} - 2a_{n-k}\sqrt{D} = \left(\sqrt{D} - a_{n-k}\right)^2 \geq 0.$$

□

The man contribution of this work is theorem 2.1, which is based on the observation that the trigonometric equation (5) can be systematically solved by an iterative method. Instead of (5) we will work with its rearranged form given by

$$\sum_{i<j}^{n-1} a_i a_j \cos(\beta_i - \beta_j) = C_n. \tag{7}$$

Moreover, a closer look into the proof of theorem 2.1 explains the motivation for the abbreviations introduced in section 2.1.

Theorem 2.1 (Computation of circular configurations)**.** *A circular configuration $\beta^{n-1} \in \mathcal{CC}_a$, that is a configuration satisfying the equation* (7) *can be obtained by the following procedure:*

1. *Compute a vector $C^{n-1} = (C_3, \dots, C_{n-1})$ with entries satisfying*

$$C_{n-k+1}^{-} \le C_{n-k} \le C_{n-k+1}^{+}, \quad \textit{and} \quad C_{n-k}^{\min} \le C_{n-k} \le C_{n-k}^{\max} \tag{8}$$

   for $1 \le k \le n-3$, where

$$C_{n-k+1}^{\pm} = C_{n-k+1} + a_{n-k}^2 \pm a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}}. \tag{9}$$

   and $C_{n-k}^{\min}, C_{n-k}^{\max}$ denote the maximal and minimal values that $X(\beta^{n-k-1})$ can take on.

2. *For $1 \le k \le n-2$ compute β_{n-k} according to the equation*

$$a_{n-k} \sin\left(\beta_{n-k} + \Phi\left(\beta^{n-k-1}\right)\right) = \frac{C_{n-k+1} - C_{n-k}}{\sqrt{S_{n-k-1} + 2C_{n-k}}}, \tag{10}$$

   whenever the denominator of the term on the right side is not zero. Otherwise the angle β_{n-k} can be chosen arbitrarily. Here we set $C_2 := 0$ and $\beta^1 := \beta_1$.

Then $\beta^{n-1} = (\beta_1, \dots, \beta_{n-1}) \in \mathcal{CC}_a$.

Proof. Assume β^{n-1} solves equation (7), can be rewritten in the following manner

$$a_{n-1} \sum_{j=1}^{n-2} a_j \cos\left(\beta_{n-1} - \beta_j\right) + \sum_{i<j}^{n-2} a_i a_j \cos(\beta_i - \beta_j) = C_n.$$

Applying the addition theorem for the cosine function and using the introduced abbreviations from section 2.1 we obtain that β^{n-1} is also a solution to

$$a_{n-1} \sin\left(\beta_{n-1}\right) \sum_{j=1}^{n-2} a_j \sin\left(\beta_j\right) + a_{n-1} \cos\left(\beta_{n-1}\right) \sum_{j=1}^{n-2} a_j \cos\left(\beta_j\right) + X\left(\beta^{n-2}\right) = C_n.$$

The latter equation is a linear combination of $\sin(\beta_{n-1})$ and $\cos(\beta_{n-1})$ and thus can further be simplified to

$$\sqrt{\left(\sum_{j=1}^{n-2} a_j \sin(\beta_j)\right)^2 + \left(\sum_{j=1}^{n-2} a_j \cos(\beta_j)\right)^2} a_{n-1} \sin\left(\beta_{n-1} + \Phi\left(\beta^{n-2}\right)\right) +$$
$$X\left(\beta^{n-2}\right) = C_n.$$

Expanding the squares shows that this is equivalent to

$$a_{n-1} \sin\left(\beta_{n-1} + \Phi\left(\beta^{n-2}\right)\right) \sqrt{S_{n-2} + 2X\left(\beta^{n-2}\right)} + X\left(\beta^{n-2}\right) = C_n,$$

where the square root is just given by the abbreviation $L(\beta^{n-2})$. If $L(\beta^{n-2}) = 0$ we have that $X(\beta^{n-2}) = C_n$ has to be satisfied and β_{n-1} is an arbitrary value. Otherwise, rewriting the latter equation gives

$$a_{n-1} \sin\left(\beta_{n-1} + \Phi\left(\beta^{n-2}\right)\right) = \frac{C_n - X\left(\beta^{n-2}\right)}{\sqrt{S_{n-2} + 2X\left(\beta^{n-2}\right)}}. \tag{11}$$

Since the latter equation can be solved for β_{n-1} the right hand side satisfies

$$-a_{n-1} \leq \frac{C_n - X\left(\beta^{n-2}\right)}{\sqrt{S_{n-2} + 2X\left(\beta^{n-2}\right)}} \leq a_{n-1},$$

which is the case when $X\left(\beta^{n-2}\right)$ is equal to a number C_{n-1} that is contained within the roots $C_n^{\pm}$ of the quadratic equation

$$\left(C_n - C\right)^2 - a_{n-1}^2\left(S_{n-2} + 2C\right) = 0, \tag{12}$$

and when $C^{\min} \leq C_{n-1} \leq C^{\max}$ holds. Note that the roots of the latter equation are real by lemma 2.1. Thus we have that β^{n-2} satisfies at the equation

$$X\left(\beta^{n-2}\right) = \sum_{i<j}^{n-2} a_i a_j \cos\left(\beta_i - \beta_j\right) = C_{n-1}, \tag{13}$$

which is of the same Type as (7). Consequently, the computations above can be repeated and after k times we end up with

$$a_{n-k} \sin\left(\beta_{n-k} + \Phi\left(\beta^{n-k-1}\right)\right) = \frac{C_{n-k+1} - X\left(\beta^{n-k-1}\right)}{\sqrt{S_{n-k-1} + 2X\left(\beta^{n-k-1}\right)}}.$$

Again, if $L(\beta^{n-k}) \neq 0$, the latter equation can be solved for the β_{n-k} and thus $X\left(\beta^{n-k-1}\right)$ equals a value C_{n-k} that satisfies $C_{n-k}^{\min} \leq C_{n-k} \leq C_{n-k}^{\max}$ and which is contained within the roots $C_{n-k+1}^{\pm}$ of the equation

$$\left(C_{n-k+1} - C\right)^2 - a_{n-k}^2\left(S_{n-k-1} + 2C\right) = 0,$$

given by (9). Thus a circular configuration $\beta^{n-1} \in \mathcal{CC}_a$ defines values C_{n-k} satisfying the system inequalities (8). Conversely, if we have s solution to systems (8) a circular configuartion $\beta^{n-1} \in \mathcal{CC}_a$ can be defined according to (10). □

The C^{n-3}, which entries C_{n-k} are recursively obtained by the system of inequalities (8) form a domain in $n-3$ dimensional real space. By the proof of the last theorem it is clear that the parameters C_{n-k} are closely related to the abbreviations $L\left(\beta^{n-k-1}\right)$ introduced in section 2.1. More precisely, for $\beta \in \mathcal{CC}_a$ the term $L(\beta^{n-k})$ is the length of the line segment connecting the origin with the endpoint $f_{a,n-k}(\beta^{n-k})$, which we will call a *diagonal* of a CKC according to Han et al. (2008*b*), see Figure 2.1. In the appendix A the relation between C_{n-k} and the diagonals of a CKC is explained in more detail. However, the connection between the C_{n-k} and the diagonals of a CKC motivates the following definition.

Definition 2.1 (Domain of Semi-diagonals). *We will denote the set given by*

$$\mathcal{SD}_a := \left\{C^{n-1} \in \mathbb{R}^{n-3} \colon C_{n-k} \in \left[C^-_{n-k+1}, C^+_{n-k+1}\right], 1 \leq k \leq n-3\right\} \tag{14}$$

as semi-diagonal parameters of a CKC with links a.

According to theorem 2.1 from any $C^{n-3} \in \mathcal{SD}_a \cap Q_a$, where

$$Q_a = \left\{C^{n-3} \in \mathbb{R}^{n-3} \colon C^{\min}_{n-k} \leq C_{n-k} \leq C^{\max}_{n-k}\right\},$$

circular configurations can be computed by solving (10). Note, that $C^{\max}_{n-k} = \sum_{i<j}^{n-k} a_i a_j$ and $C^{\min}_{n-k} = \min_\beta X(\beta^{n-k-1})$ can be easily computed. Since the solution of (10) for a β_{n-k} is not unique, an element $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ will yield several circular configurations and we will consider all possible configurations that can be obtained from it in section 3.2. In the following section we will further investigate the set $\mathcal{SD}_a$. It will turn out, that it can be described in a very easy way, after a substitution of variables and thus also leads to an easy description for $\mathcal{SD}_a \cap Q_a$.

3 Further analysis of circular configurations

In this section we will study the domain $\mathcal{SD}_a$, which naturally appears when we compute circular configurations, in more detail. Moreover, we will have a closer look on the second step of theorem 2.1, which requires solving equation (10) for an angle β_{n-k}.

3.1 The domain of semi-diagonals

We investigate the parameter space $\mathcal{SD}_a$ further. We recall that for a CKC with n links $a_1, a_2, \ldots, a_n$ the set $\mathcal{SD}_a$ is defined by the tuples C^{n-3}, which entries satisfy the system

$$C^-_{n-k+1} \leq C_{n-k} \leq C^+_{n-k+1} \tag{15}$$

of inequalities, where

$$C^{\pm}_{n-k+1} = C_{n-k+1} + a^2_{n-k} \pm a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}}$$

for $1 \leq k \leq n-3$. We will see that this system can be transformed into another system after a substitution of variables, which can then be easily parametrized by a map that is defined on the $n-3$ dimensional unit cube $I^{n-3} = [-1,1]^{n-3}$.

Theorem 3.1. *Let $C^{n-1} \in \mathcal{SD}_a$ and define new parameters U_{n-k} by*

$$C_{n-k} = U_{n-k} + C_{n-k+1} + a_{n-k}^2, \tag{16}$$

for $1 \le k \le n-3$. Moreover we set $U_n := C_n$ and $U^{n-1} = (U_3, \ldots, U_{n-1})$. Then the entries U_{n-k} satisfy the new system of inequalities

$$-a_{n-k}\sqrt{t_k\left(U_n, \ldots, U_{n-k+1}\right)} \le U_{n-k} \le a_{n-k}\sqrt{t_k\left(U_n, \ldots, U_{n-k+1}\right)}, \tag{17}$$

where

$$t_k\left(U_n, \ldots, U_{n-k+1}\right) = 2\sum_{j=1}^{k-1} U_{n-j} + \sum_{j=0}^{k-1} a_{n-j}^2, \text{ for } 1 \le k \le n-3. \tag{18}$$

Note that the first sum on the left is zero in the case $k = 1$ and thus $t_1 = t_1\left(U_n\right) = a_n^2$.

Proof. We will apply (16) to the right side of system (15) only, since the computations for the left side are analogous. If we apply (16) we obtain

$$U_{n-k} + C_{n-k+1} + a_{n-k}^2 \le C_{n-k+1} + a_{n-k}^2 \pm a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}},$$

which is equivalent to

$$U_{n-k} \le a_{n-k}\sqrt{2C_{n-k+1} + S_{n-k}}.$$

If we apply the substitution (16) repeatedly on the right side we end up with

$$\begin{aligned} U_{n-k} &\le a_{n-k}\sqrt{2\left(U_n + \sum_{j=1}^{k-1} U_{n-j} + \sum_{j=1}^{k-1} a_{n-j}^2\right) + S_{n-k}} \\ &\le a_{n-k}\sqrt{2U_n + 2\sum_{j=1}^{k-1} U_{n-j} + \sum_{j=1}^{k-1} a_{n-j}^2 + S_{n-1}} \\ &\le a_{n-k}\sqrt{2\sum_{j=1}^{k-1} U_{n-j} + \sum_{j=0}^{k-1} a_{n-j}^2}, \end{aligned}$$

where the expression under the square root is the abbreviation $t_k\left(U_n, \ldots, U_{n-k+1}\right)$ in (18). □

Let $\mathcal{SDU}_a$ denote the space of theorem 3.1. It is clear that any $U^{n-1} \in \mathcal{SDU}_a$ yields a unique $C^{n-1} \in \mathcal{SD}_a$ and vice versa since (16) defines an affine map from $\mathrm{A}_a \colon \mathcal{SDU}_a \to \mathcal{SD}_a$. We will further investigate the domain $\mathcal{SDU}_a$ defined by the system of inequalities (17), which we can explicitly describe by a map defined on the unit cube.

Theorem 3.2. *Let I^{n-3} be the $n-3$ dimensional unit cube and define the k-tuple $s^k := (s_1, \ldots, s_k)$ for $1 \le k \le n-3$. Then*

$$\mathrm{P}^a \colon I \longrightarrow \mathcal{SDU}_a, \quad s^{n-3} \mapsto \Phi^a\left(s^{n-3}\right),$$

where the components of P^a *are given by*

$$\begin{aligned}
\mathrm{P}^a_3\left(s^{n-3}\right) &= s_{n-3}a_3\sqrt{t_{n-3}\left(U_n,\mathrm{P}^a_{n-1}\left(s^1\right),\dots,\mathrm{P}^a_4\left(s^{n-4}\right)\right)}\\
&\vdots\\
\mathrm{P}^a_{n-2}\left(s^2\right) &= s_2a_{n-2}\sqrt{t_2\left(U_n,\mathrm{P}^a_{n-1}\left(s^1\right)\right)}\\
\mathrm{P}^a_{n-1}\left(s^1\right) &= s_1a_{n-1}\underbrace{\sqrt{t_1\left(U_n\right)}}_{a_n}
\end{aligned}$$

maps the unit cube onto the space $\mathcal{SDU}_a$.

Proof. We have to show that $U_{n-k} := \mathrm{P}^a_k\left(s^{n-k}\right)$ satisfy the system of inequalities (17). We have that

$$-a_{n-k}\sqrt{t_k\left(U_n,\dots,U_{n-k+1}\right)} \leq U_{n-k} \leq a_{n-k}\sqrt{t_k\left(U_n,\dots,U_{n-k+1}\right)}.$$

Note that the left an right sides of these inequalities do not depend on U_{n-k}. Thus plugging in

$$\mathrm{P}^a_{n-k}\left(s^k\right) = s_ka_{n-k}\sqrt{t_k\left(U_n,\dots,\mathrm{P}^a_{n-k+1}\left(s^{n-k+1}\right)\right)}$$

we obtain for the right side of the latter system

$$s_ka_{n-k}\sqrt{t_k\left(U_n,\dots,\mathrm{P}^a_{n-k+1}\left(s^{n-k+1}\right)\right)} \leq a_{n-k}\sqrt{t_k\left(U_n,\dots,\mathrm{P}^a_{n-k+1}\left(s^{n-k+1}\right)\right)}$$

and for its left side

$$-a_{n-k}\sqrt{t_k\left(U_n,\dots,\mathrm{P}^a_{n-k+1}\left(s^{n-k+1}\right)\right)} \leq s_ka_{n-k}\sqrt{t_k\left(U_n,\dots,\mathrm{P}^a_{n-k+1}\left(s^{n-k+1}\right)\right)}.$$

Clearly, both inequalities are satisfied if $s_k \in [-1,1]$ and thus system 17 is satisfied and thus we have that $\mathrm{P}^a\left(I^{n-3}\right) = \mathcal{SDU}_a$. □

Remark 3.1 (Illustrative example)**.** In order to better understand the statement of theorem we demonstrate it for the CKC with equal links $a_i = 1$ for $1 \leq i \leq 5$. In this case the inequalities defining $\mathcal{SDU}_a$ are given by

$$\begin{aligned}
-1 \leq& U_4 \leq 1\\
\sqrt{2U_4+2} \leq& U_3 \leq \sqrt{2U_4+2}
\end{aligned}$$

Then the desired map is given by

$$\mathrm{P}^a\colon\ [-1,1]^2 \to \mathcal{SDU}_a, \quad \begin{pmatrix} s_1\\ s_2\end{pmatrix} \mapsto \begin{pmatrix} s_1\\ s_2\sqrt{2s_1+2}\end{pmatrix}.$$

Note that P^a is injective for $s_1 \neq -1$.

The map P^a gives an easy description for parameters U_{n-k} satisfying the system of inequalities (17). Thus using the affine transform A_a we obtain a map $A_a \circ \mathrm{P}^a\colon I^{n-3} \to \mathcal{SD}_a$, which can be used to easily obtain points in $\mathcal{SDU}_a \cap Q_a$, when restrictions $C^{\min}_{n-k} \leq C_{n-k} \leq C^{\max}_{n-k}$ are taken into account. Further investigations of P^a like its injectivity or its singularities are an interesting topic for future research.

3.2 Flipping over lines through diagonals of a CKC

Given a $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ the angle β_{n-k} is computed from

$$a_{n-k} \sin\left(\beta_{n-k} + \Phi\left(\beta^{n-k-1}\right)\right) = \frac{C_{n-k+1} - X\left(\beta^{n-k-1}\right)}{\sqrt{S_{n-k-1} + 2X\left(\beta^{n-k-1}\right)}}.$$

according to theorem 2.1. Clearly, solving for β_{n-k} is not unique. Each time we solve for β_{n-k} we have to choose which pre image we will take. In the following let $\varepsilon^k = (\varepsilon_2, \dots, \varepsilon_k) \in \{0,1\}^{k-1}$ be a vector that contains the information, which pre images haven been chosen. We will refer to ε^k as orientation vector. More precisely we set,

$$\beta_{n-k}^{\varepsilon_{n-k}} = \pi\varepsilon_{n-k} + (-1)^{\varepsilon_{n-k}} S_{n,k}\left(C^{n-1}\right) - \Phi\left(\beta^{n-k-1,\varepsilon^{n-k-1}}\right). \tag{19}$$

for $1 \le k \le n-2$. Here the superscript ε_{n-k} in $\beta_{n-k}^{\varepsilon_{n-k}}$ indicates, which preimage is chosen according to equation (19). Note, that we used the abbreviation

$$S_{n,k}\left(C^{n-1}\right) = \arcsin\left(\frac{C_{n-k+1} - C_{n-k}}{a_{n-k}\sqrt{S_{n-k-1} + 2C_{n-k}}}\right)$$

for $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ and the notation

$$\beta^{n-k,\varepsilon^{n-k}} = \left(\beta_1, \beta_2^{\varepsilon_2}, \dots, \beta_{n-k}^{\varepsilon_{n-k}}\right),$$

to indicate the choice of preimages.

Therefore, for each $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ and a $\varepsilon \in \{0,1\}^{n-2}$ we obtain a circular configuration

$$\beta^{n-1,\varepsilon^{n-1}} = \left(\beta_1, \beta_2^{\varepsilon_2}, \dots, \beta_{n-1}^{\varepsilon_{n-1}}\right)$$

by formula (19). Note that the angle β_1 does note have a superscript since β_1 is chosen arbitrarily in the last step of (10) for $k = n-3$. Each $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ yields 2^{n-2} circular configurations which corresponds to the possible choices for the components of ε^{n-1}. There is a geometric interpretation for the value ε_{n-k} in equation (19). It describes how the link a_{n-k} is flipped over the line running through the origin and $f\left(\beta^{n-k-1,\varepsilon^{n-k-1}}\right)$, when it is attached to $f\left(\beta^{n-k-1,\varepsilon^{n-k-1}}\right)$, see Figure 3.2. Choosing the value ε_{n-k} corresponds to the choice of a triangle orientation in Han et al. (2008*a*) when building up a CKC from its diagonal lengths.

4 Numerical simulations

In this section we provide numerical examples that demonstrate validity of the methods developed in this work. For illustrative purposes we will consider CKCs with five and six links. For CKSs with five links we will give depict $\mathcal{SD}_a$. Moreover, we will compute random circular configurations for this CKCs by theorem 2.1 and depict them.

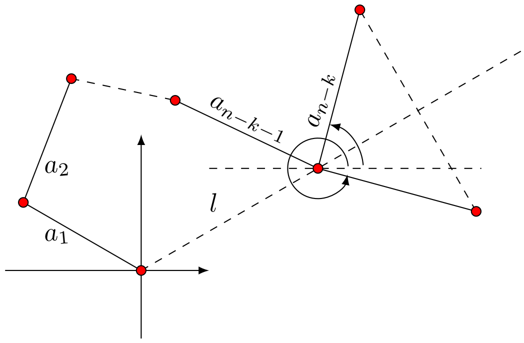

Figure 3: Choosing the value ε_{n-k} in equation (19) corresponds to a flipping over the diagonal l.

4.1 CKCs with five and six links

First we will consider a CKC with $n = 5$ and $n = 6$ links. The domain $\mathcal{SD}_a \cap Q_a$ is depicted for two CKCs with different link lengths in Figure 4. The black lines indicate the condition $-a_1a_2 = C_3^{\min} \leq C_3 \leq C_3^{\max} = a_1a_2$. The value C_4 lies within $\left[\max\left\{C_4^{\min}, C_5^-\right\}, \min\left\{C_4^{\max}, C_5^+\right\}\right]$. In the depicted cases $\max\left\{C_4^{\min}, C_5^-\right\} = C_5^-$ and $\min\left\{C_4^{\max}, C_5^+\right\} = C_5^+$ holds. Figure 5 shows 10 random circular configurations for CKCs with five links. Thereby the lengths of the last link is equal to the radius of the depicted circle. The configurations have been obtained for the orientation vector $\varepsilon^4 = (\varepsilon_2, \varepsilon_3, \varepsilon_4) = (0, 0, 0)$.

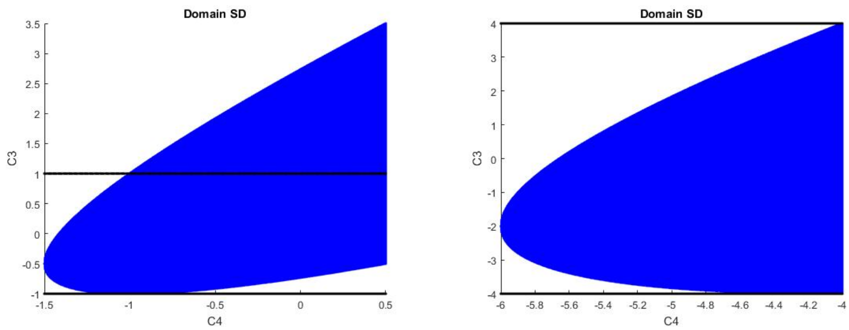

Figure 4: Domain $\mathcal{SD}_a \cap Q_a$ is the part of the blue area enclosed within the black lines. Left: CKC with all links equal to one. Right: CKC with link lengths $2, 2, 2, 1, 1$.

Finally, we give examples for CKSs with six links. Figure 6 shows random configurations for CKCs with six links. We consider the orientation vector $\varepsilon^5 = (\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (0, 0, 0, 0)$.

5 Conclusion and future work

We have developed a new method to compute configurations in terms of joint angles of a CKC by a systematic procedure. Our approach does not require

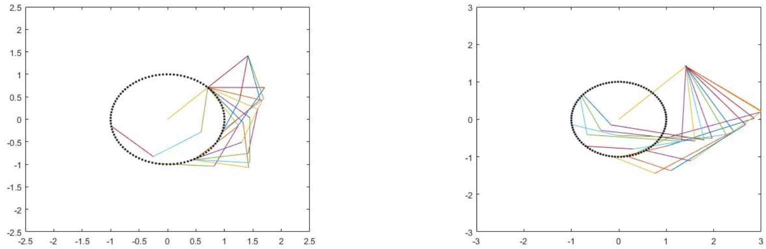

Figure 5: Circular configurations of two CKCs with five links. Left: Ten random circular configurations are depicted for the CKC with link lengths equal to one. Right: Ten random circular configurations are depicted for the CKC with link lengths $2, 2, 2, 1, 1$.

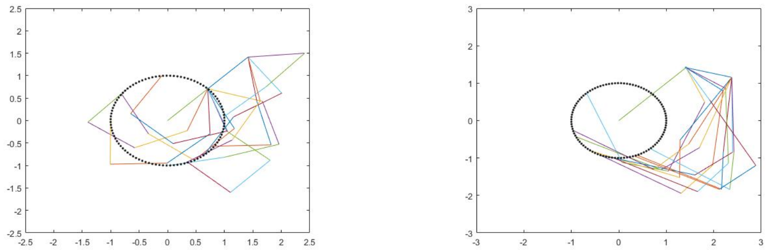

Figure 6: Circular configurations of two CKCs with six links. Left: Ten random circular configurations are depicted for the CKC with link lengths equal to one. Right: Ten random circular configurations are depicted for the CKC with link lengths $2, 1, 2, 1, 2, 1$.

the solution of a system of linear inequalities by linear programming nor does it rely on probabilist methods. Numerical examples show validity of the proposed work. We expect that the described method can be useful in tasks like motion planning for CKCs. We expect that is an interesting line for future work to investigate the introduced map $\mathrm{P}^a : I^{n-3} \to \mathcal{SDU}_a$. We expect that it enables us to use tools from differential geometry, which may lead to further insights and applications. Moreover, it would be interesting to investigate how special designs for CKCs are reflected in the presented computations.

A Appendix

As mentioned in section 2 the diagonal lengths of a CKC are closely related to the parameters C_{n-k}, which justifies the naming for $\mathcal{SD}_a$. More precisely, in Han et al. (2008*a*) it is shown that $\beta^{n-1} \in \mathcal{CC}_a$ is a circular configuration if and only its diagonal lengths satisfy the system inequalities

$$\left(L\left(\beta^{n-k}\right) - a_{n-k}\right)^2 \leq L\left(\beta^{n-k-1}\right)^2 \leq \left(L\left(\beta^{n-k}\right) + a_{n-k}\right)^2, \tag{20}$$

for $1 \leq k \leq n-2$. Note that here $L\left(\beta^1\right) = a_1$ and $L\left(\beta^{n-1}\right) = a_n$. Furthermore, it is fine here to consider circular configurations, since the diagonal lengths of a CKC are invariant with respect to rotation around the origin. The relation between the diagonal lengths an $\mathcal{SD}_a$ is established by the lemma.

Lemma A.1. *Let $\beta^{n-1} \in \left(S^1\right)^{n-1}$ be a vector of joint angles and let $C^{n-1} = (C_3, \ldots, C_{n-1})$ be a vector, which entries are given by*

$$C_{n-k} := X\left(\beta^{n-k-1}\right) \tag{21}$$

for $1 \leq k \leq n-3$. Then $\beta^{n-1} \in \mathcal{CC}_a$ if and only if $C^{n-1} \in \mathcal{SD}_a \cap Q_a$.

Proof. Assume $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ and $X\left(\beta^{n-k-1}\right) = C_{n-k}$ for a $\beta^{n-1} \in \left(S^1\right)^{n-1}$. Then, by (8) we have that the inequality $C^-_{n-k+1} \leq C_{n-k}$ is satisfied. Using the explicit expression (9) for C^-_{n-k+1} and our assumption we obtain that this is equivalent to

$$C_{n-k+1} + a^2_{n-k} - a_{n-k} L\left(\beta^{n-k}\right) \leq C_{n-k}. \tag{22}$$

Multiplying this inequality by two and then adding S_{n-k-1} on both sides gives

$$2C_{n-k+1} + 2a^2_{n-k} - 2a_{n-k} L\left(\beta^{n-k}\right) + S_{n-k-1} \leq 2C_{n-k} + S_{n-k-1} \tag{23}$$

and therefore, since $S_{n-k-1} + a^2_{n-k} = S_{n-k}$, we have that

$$\underbrace{2C_{n-k+1} + S_{n-k}}_{=L(\beta^{n-k})^2} + a^2_{n-k} - 2a_{n-k} L\left(\beta^{n-k}\right) \leq L\left(\beta^{n-k-1}\right)^2.$$

Completing the square

$$\left(L\left(\beta^{n-k}\right) - a_{n-k}\right)^2 \leq L\left(\beta^{n-k-1}\right)^2,$$

shows that (22) is equivalent to the first inequality in (20). The second one follows from $C_{n-k} \leq C^+_{n-k+1}$ by analogous computations. Thus the system of inequalities (20) is satisfied and $\beta^{n-1} \in \mathcal{CC}_a$. Conversely, if $\beta^{n-1} \in \mathcal{CC}_a$ is a circular configuration of a CKC its diagonals satisfy inequalities (20). Setting $C_{n-k} := X\left(\beta^{n-k-1}\right)$ and repeating the latter estimates shows that (8) are satisfied and thus $C^{n-1} \in \mathcal{SD}_a \cap Q_a$. □

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