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Abstract

In many tomographic applications and elsewhere, there arises a need to reconstruct a function from the data on the boundary of some domain given either by the spherical means of the function, or by the corresponding solution of the free-space wave equation. In this paper, we show that the so-called universal back-projection formulas provide exact recovery of the unknown function for data on any quadric hypersurfaces that can be approximated by elliptic hypersurfaces. These quadric hypersurfaces include elliptic paraboloid as well as parabolic and elliptic cylinders.

Keywords. Spherical means, wave equation, Radon transform, computed tomography, inversion formula, universal backprojection.

1. Introduction

Consider a convex domain $\Omega \subset \mathbb{R}^d$, where $d \geq 2$ denotes the spatial dimension, with smooth boundary $\partial\Omega$. Let $C_c^\infty(\Omega)$ denote the set of all real valued smooth functions $f \colon \mathbb{R}^d \to \mathbb{R}$ that are compactly supported in Ω. In the present paper, we deal with the problem of reconstructing an unknown function $f \in C_c^\infty(\Omega)$ from the data on the boundary $\partial\Omega$, which either consists of the spherical means of f or the solution of the standard free-space wave equation with initial data $(f, 0)$. In particular, we investigate the universal back-projection formula (see [1, 2, 3, 4, 5, 6]) on quadric hypersurfaces that can be approximated by elliptic hypersurfaces. For such type of quadrics we will show that the universal back-projection formula provides an exact reconstruction.

1.1. Inversion from spherical means and the wave equation

We consider the spherical means operator $\mathcal{M} \colon C_c^\infty(\Omega) \to C^\infty(\mathbb{R}^n \times (0,\infty))$ that maps a function $f \in C_c^\infty(\Omega)$ to the spherical means $\mathcal{M}f \colon \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ defined by

$$(\mathcal{M}f)(x,r) := \frac{1}{\omega_{d-1}} \int_{S^{d-1}} f(x+ry)\,\mathrm{d}s(y), \qquad \text{for } (x,r) \in \mathbb{R}^d \times (0,\infty)\,.$$

Here $S^{d-1} \subset \mathbb{R}^d$ is the $(d-1)$-dimensional unit sphere, ω_{d-1} is its total surface area, and $\mathrm{d}s$ denotes the standard surface measure. We also consider the solution operator $\mathcal{W} \colon f \mapsto p$

of the standard free-space wave equation, that maps a function $f \in C_c^\infty(\Omega)$ to the solution $p\colon \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ of the following initial value problem

$$\begin{cases} (\partial_t^2 - \Delta_x)\, p(x,t) = 0 & \text{for } (x,t) \in \mathbb{R}^d \times (0,\infty), \\ p(x,0) = f(x) & \text{for } x \in \mathbb{R}^d, \\ (\partial_t p)(x,0) = 0 & \text{for } x \in \mathbb{R}^d\,. \end{cases} \tag{1.1}$$

With the operators just introduced, the considered reconstruction problems can be formulated as follows: recover the unknown function $f \in C_c^\infty(\Omega)$ from either its restricted spherical means

$$m(x,r) = (\mathcal{M}f)\,(x,r) \quad \text{for } (x,r) \in \partial\Omega \times (0,\infty)\,, \tag{1.2}$$

or the corresponding wave data

$$p(x,t) = (\mathcal{W}f)\,(x,t) \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty)\,. \tag{1.3}$$

These two problems are essentially equivalent because the solution operator $\mathcal{W}$ can be expressed through the spherical means operator $\mathcal{M}$, and vice versa.

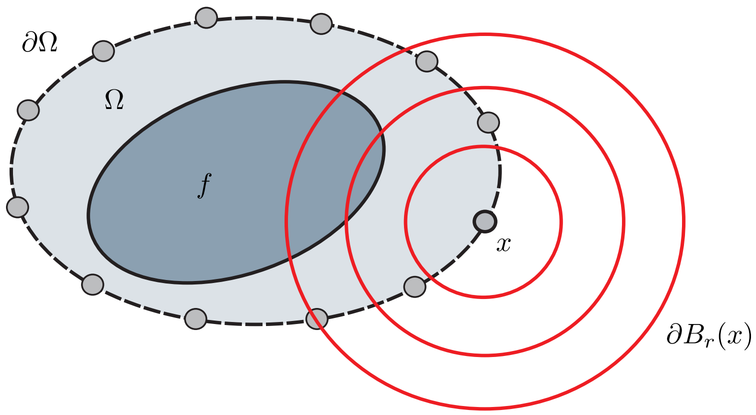

Figure 1: *Illustration of the reconstruction problems.* The function f is supported inside the domain Ω. Detectors are placed on the boundary $\partial\Omega$ of the domain and record either averages of f over spherical surfaces centered at $\partial\Omega$ or the solution of the wave equation restricted to $\partial\Omega \times (0,\infty)$.

The problems of recovering a function from the spherical means (1.2) or the wave data (1.3) arise, for example, in the so-called photoacoustic tomography (PAT) and thermoacoustic tomography (TAT), where the unknown function f represents the initial pressure of an ultrasonic wave that is induced by a short electromagnetic pulse. In PAT/TAT using point-like detectors these problem arise in three spatial dimensions (see, for example, [7, 8, 9]). When using linear of circular integrating detectors in PAT/TAT, these reconstruction problems arise in two spatial dimensions (see [10, 11, 12, 13]). The problems of recovering a function from its spherical means of the wave data also arises in other technologies, such as SONAR [14, 15], SAR imaging [16, 17, 18], ultrasound tomography [19, 20], and seismic imaging [21, 22].

The derivation of explicit inversion formulas for recovering a function from data (1.2) or (1.3) has recently been addressed by many authors. Such formulas are currently only known for special boundaries $\partial\Omega$. For example, explicit inversion formulas have been derived for hyperplanes [1, 16, 22, 23, 24, 25], spheres [1, 2, 26, 27, 28] and cylinders [20]. Reconstruction formulas for some polygons and polyhedra in two and three spatial dimensions have been obtained in [29]. Recently, explicit formulas for inverting (1.2) and (1.3) on elliptical domains have been derived in [3, 4, 6, 30, 31, 32]. For the special case of spherical domains, the formulas in [3, 4, 6] reduce the ones earlier derived in [1, 2]. According to the notion of [1] we call these formulas the universal back-projection formulas.

It should be noted that in [3, 4, 5, 6] the use of the universal back-projection formula on general convex bounded domains Ω was analyzed. As a result, on a general convex bounded domain, the universal back-projection formula recovers the unknown function f up to an additional error term that has been explicitly computed in [3, 4, 6]. For elliptic domains, in [3, 4, 6] the error term has been shown to vanish identically. In [5], the error term has been analyzed from a microlocal point of view. One of the results of [5] shows that for parabolic domains the error term is an infinitely smoothing operator. Further, according to [5], one can show that the error term is an infinitely smoothing operator for any domain of the form

$$\Omega = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{b} \bar{\alpha}_i x_i^2 < \sum_{i=b+1}^{d} \bar{\alpha}_i x_i^2 \; + \; \bar{\alpha}_{d+1} \right\}, \tag{1.4}$$

where $x = (x_1, x_2, \dots, x_d)$, $b \in \{1, \dots, d\}$, and $\bar{\alpha}_i \geq 0$ with $(\bar{\alpha}_1, \dots, \bar{\alpha}_b), (\bar{\alpha}_{b+1}, \dots, \bar{\alpha}_d) \neq 0$. After an appropriate affine transformation, any domain of the form (1.4) belongs to one of the following classes:

- elliptic domains
$$E = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} \alpha_i x_i^2 < 1 \right\}, \tag{1.5}$$
- elliptic paraboloids
$$P = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d-1} \alpha_i x_i^2 < x_d \right\}, \tag{1.6}$$
- parabolic cylinders
$$C_{\text{par}} = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{b} \alpha_i x_i^2 < x_d \right\} \text{ with } 1 \leq b \leq d-2, \tag{1.7}$$
- elliptic cylinders
$$C_{\text{ell}} = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{b} \alpha_i x_i^2 < 1 \right\} \text{ with } 1 \leq b \leq d-1. \tag{1.8}$$

In the definitions of any of these domains, all coefficients α_i are supposed to be positive.

The fact that the error term is an infinitely smoothing operator for any domain of the form (1.4) indicates that the universal back-projection formula may be also exact for these domains. In this paper, we prove that this is indeed the case. One of the key observations our proof is the possibility of approximating any domain of the form (1.4) by elliptic domains (1.5). We also adapt the techniques of [33], where we showed that the universal back-projection is exact for parabolic domains in two spatial dimensions.

1.2. Statement of main results

First, let us present the universal back-projection formulas $\mathcal{F}_d$ and $\mathcal{G}_d$ for the spherical means m and wave data p, respectively, as presented in [6] (see also [1, 2, 3, 4]). In addition to the data, the universal back-projection formulas also depend on the boundary $\partial\Omega$ of the domain

$\Omega \subset \mathbb{R}^d$ and on the reconstruction point $x_0 \in \Omega$. The structure of the formulas further depends on whether the spatial dimension d is even or odd.

If $d \geq 2$ is an even integer, then

$$\mathcal{F}_d(\partial\Omega, m, x_0) := \kappa_d^{(1)} \int_{\partial\Omega} \langle \nu_x, x_0 - x \rangle \int_0^\infty \frac{\left(\partial_r \mathcal{D}_r^{d-2} r^{d-2} m\right)(x, r)}{r^2 - |x_0 - x|^2} \,\mathrm{d}r \,\mathrm{d}s(x)\,, \tag{1.9}$$

$$\mathcal{G}_d(\partial\Omega, p, x_0) := \kappa_d^{(2)} \int_{\partial\Omega} \langle \nu_x, x_0 - x \rangle \int_{|x_0 - x|}^\infty \frac{\left(\partial_t \mathcal{D}_t^{(d-2)/2} t^{-1} p\right)(x, t)}{\sqrt{t^2 - |x_0 - x|^2}} \,\mathrm{d}t \,\mathrm{d}s(x)\,. \tag{1.10}$$

Here $\kappa_d^{(1)} := (-1)^{(d-2)/2} \omega_{d-1} / \left(2\pi^d\right)$ and $\kappa_d^{(2)} := (-1)^{(d-2)/2} / \pi^{d/2}$ are some constants, ν_x denotes the outward pointing unit normal to $\partial\Omega$, and $\mathcal{D}_r := (2r)^{-1}\partial_r$ is the differentiation operator with respect to r^2. Further, $\langle \,\cdot\,,\,\cdot\,\rangle$ and $|\cdot|$ denote the standard inner product and the corresponding Euclidian norm on $\mathbb{R}^d$, respectively. The inner integral in (1.9) is understood in the principal value sense.

In the case of odd dimension $d \geq 3$, the universal back-projection formulas $\mathcal{F}_d$ and $\mathcal{G}_d$ are defined as follows:

$$\mathcal{F}_d(\partial\Omega, m, x_0) := \kappa_d^{(1)} \int_{\partial\Omega} \frac{\langle \nu_x, x_0 - x \rangle}{|x_0 - x|} \left(\partial_r \mathcal{D}_r^{d-2} r^{d-2} m\right)(x, |x_0 - x|) \,\mathrm{d}s(x)\,, \tag{1.11}$$

$$\mathcal{G}_d(\partial\Omega, p, x_0) := \kappa_d^{(2)} \int_{\partial\Omega} \frac{\langle \nu_x, x_0 - x \rangle}{|x_0 - x|} \left(\partial_t \mathcal{D}_t^{(d-3)/2} t^{-1} p\right)(x, |x_0 - x|) \,\mathrm{d}s(x)\,, \tag{1.12}$$

with constants $\kappa_d^{(1)} := (-1)^{(d-3)/2} \omega_{d-1} / (4\pi^{d-1})$ and $\kappa_d^{(2)} := (-1)^{(d-3)/2} / (2\pi^{(d-1)/2})$.

In [6], it has been shown that for any elliptic domain the formulas $\mathcal{F}_d$ and $\mathcal{G}_d$ provide exact reconstruction of a function f from the spherical means data (1.2) and the wave data (1.3), respectively. Let us specify mathematically this result.

Theorem 1.1 (Universal back-projection for elliptical domains, see [6])**.** *Let $E \subset \mathbb{R}^d$, with $d \geq 2$, be an elliptic domain of the form (1.5) and let $f \in C_c^\infty(E)$. Then, for every reconstruction point $x_0 \in E$,*

$$f(x_0) = \mathcal{F}_d(\partial E, \mathcal{M}f, x_0) = \mathcal{G}_d(\partial E, \mathcal{W}f, x_0)\,,$$

where $\mathcal{F}_d$, $\mathcal{G}_d$ are defined by (1.9), (1.10) and (1.11), (1.12) for even and odd d, respectively.

In this paper, we will show that the above theorem holds if E is replaced by any domain of the form (1.4). Precisely speaking, we will show the following theorems.

Theorem 1.2 (Inversion from spherical means)**.** *Let $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, be a domain of the form (1.4). Also, let $f \in C_c^\infty(\Omega)$ and define $\mathcal{F}_d$ by (1.9) and (1.11) for even and odd d, respectively. Then, for every reconstruction point $x_0 \in \Omega$,*

$$f(x_0) = \mathcal{F}_d(\partial\Omega, \mathcal{M}f, x_0)\,. \tag{1.13}$$

In particular, the integral in $\mathcal{F}_d(\partial\Omega, \mathcal{M}f, x_0)$ is absolutely convergent.

Proof. See Section 3.2. □

Theorem 1.3 (Inversion of the wave equation)**.** *Let $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, be a domain of the form (1.4). Also, let $f \in C_c^\infty(\Omega)$, and $\mathcal{G}_d$ be defined by (1.10) and (1.12) for even and odd d, respectively. Then, for every reconstruction point $x_0 \in \Omega$,*

$$f(x_0) = \mathcal{G}_d(\partial\Omega, \mathcal{W}f, x_0)\,. \tag{1.14}$$

In particular, the integral in $\mathcal{G}_d(\partial\Omega, \mathcal{W}f, x_0)$ is absolutely convergent.

Proof. See Section 3.3. □

As we already noted, any domains of the form (1.4) can be approximated by elliptic domains. Roughly speaking, we will show that for a domain Ω in (1.5), there is a sequence of elliptic domains E_n such that the sequence ∂E_n converges pointwise almost everywhere to the boundary $\partial\Omega$. Then, the proof of Theorem 1.2 is based on the corresponding results for elliptical domains and the dominated convergence theorem. For application of the dominated convergence, we derive equivalent representations of formulas (1.9) and (1.11). Theorem 1.3 follows from Theorem 1.2 by exploiting relationships between the operators $\mathcal{M}$ and $\mathcal{W}$.

1.3. Organization of the paper

The rest of this paper is organized as follows. In Section 2, we derive auxiliary results that we require for the proofs of Theorems 1.2 and 1.3. In particular, we derive equivalent representations of the formulas (1.9) and (1.11) that we will use for the application of the dominated convergence theorem in the proof of Theorem 1.2. Further, we will study the approximation of domains (1.4) by elliptic domains. We present the proofs of Theorems 1.2 and 1.3 in Section 3. The paper concludes with some discussions in Section 4.

2. Auxiliary results

In this section we derive auxiliary propositions that will be used for the proof of our main results. The first auxiliary result is an alternative representation of the inversion integral $\mathcal{F}_d$ given by (1.9), (1.11). These representations are given in Propositions 2.1 and 2.2. The second auxiliary result concerns the approximation of domains of the form (1.4) by elliptic domains; see Proposition 2.4.

2.1. Alternative representations of $\mathcal{F}_d$

In this section, we derive equivalent expressions for $\mathcal{F}_d$ by transforming the integrals over $\partial\Omega$ in (1.9) and (1.11) into integrals over the unit sphere S^{d-1}. For this purpose, we construct a special parameterization $\Phi\colon S^{d-1} \to \partial\Omega$ of the boundary $\partial\Omega$ depending on the reconstruction point $x_0 \in \Omega$.

For any given reconstruction point $x_0 \in \Omega$ we define the mapping

$$\Psi\colon \partial\Omega \to S^{d-1}\colon x \mapsto \frac{x - x_0}{|x - x_0|}\,.$$

If Ω is convex and bounded, then Ψ is bijective and therefore, invertible. We call the corresponding inverse function

$$\Phi\colon S^{d-1} \to \partial\Omega\colon y \mapsto \Psi^{-1}(y) \tag{2.1}$$

the spherical parameterization of $\partial\Omega$ around the point $x_0 \in \Omega$. It can be easily seen, that $\Phi(y)$ for $y \in S^{d-1}$ is the unique element in the intersection of the ray $\{x_0 + ty \mid t > 0\}$ with the boundary $\partial\Omega$, see Figure 2. Note that the spherical parametrisation Φ also depends on x_0. In order to keep the notation simple, we do not indicate this dependance.

For unbounded convex domains Ω, such as (1.4), the image of the function Ψ does not contain a certain subset $S \subset S^{d-1}$. In such a situation, the corresponding spherical parameterization $\Phi\colon S^{d-1} \setminus S \to \partial\Omega$ is defined only for $y \notin S$. For domains (1.4), this set S is actually a set of surface measure zero, compare with Remark 2.3 below.

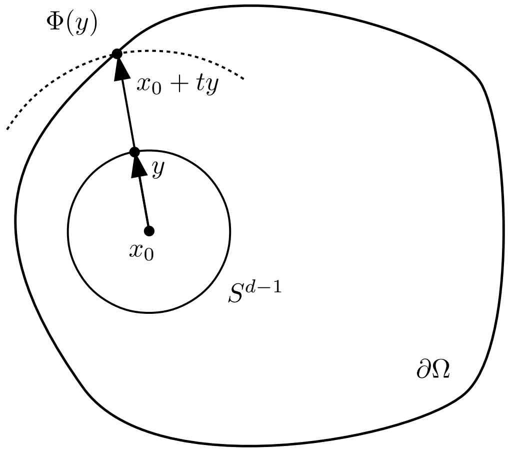

Figure 2: Illustration of the spherical parameterization Φ defined in (2.1) of the boundary $\partial\Omega$ around the point $x_0 \in \Omega$. For any $y \in S^{d-1}$ we have $\{\Phi(y)\} = \partial\Omega \cap \{x_0 + ty \mid t > 0\}$.

Let Ω be convex and bounded. Then, using the spherical parameterization and the transformation rule, for any measurable function $g\colon \partial\Omega \to \mathbb{R}$ we have

$$\int_{\partial\Omega} g(x)\, \langle \nu_x, x - x_0\rangle \,\mathrm{d}s(x) = \int_{S^{d-1}} g\left(\Phi(y)\right) \left|\Phi(y) - x_0\right|^d \mathrm{d}s(y)\,. \tag{2.2}$$

Based on the integral identity (2.2) we can derive equivalent expressions for the formula $\mathcal{F}_d$ that involve an integral over the unit sphere S^{d-1}. These expressions are presented in the next proposition.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a convex bounded domain, $f \in C_{\mathrm{c}}^\infty(\Omega)$, and $x_0 \in \Omega$. Further, set $m(x,r) := (\mathcal{M}f)(x,r)$ and $(\mathcal{B}m)(x,r) := (\mathcal{D}_r^{d-2} r^{d-2} m)(x,r)$. Then, with the spherical parameterization Φ defined in (2.1), the formula $\mathcal{F}_d$ defined by (1.9), (1.11) can be expressed as follows:*

(a) If $d \geq 2$ is even, then

$$\mathcal{F}_d(\partial\Omega, m, x_0) = -\kappa_d^{(1)} \int_{S^{d-1}} |x_0 - \Phi(y)|^d \int_0^\infty \frac{\partial_r \mathcal{B}m\left(\Phi(y), r\right)}{r^2 - |x_0 - \Phi(y)|^2} \,\mathrm{d}r\mathrm{d}s(y)\,. \tag{2.3}$$

(b) If $d \geq 3$ is odd, then

$$\mathcal{F}_d(\partial\Omega, m, x_0) = -\kappa_d^{(1)} \int_{S^{d-1}} |x_0 - \Phi(y)|^{d-1} \partial_r \mathcal{B}m\left(\Phi(y), |x_0 - \Phi(y)|\right) \mathrm{d}s(y)\,. \tag{2.4}$$

Proof. This immediately follows from the expressions (1.9), (1.11) for $\mathcal{F}_d$, the definitions of m and $\mathcal{B}m$, and the integral identity (2.2). $\square$

In the case of even spatial dimensions, (2.3) can be further modified such that it involves an integral over a set with a finite measure being independent of $\partial\Omega$. Such an expression will be used for the application of the dominated convergence theorem in the proof of Theorem 1.2.

Proposition 2.2. *Let $d \geq 2$ be even, $\Omega \subset \mathbb{R}^d$ be a convex bounded domain, $f \in C_{\mathrm{c}}^\infty(\Omega)$, $x_0 \in \Omega$, and let Φ be the spherical parameterization around x_0 defined by (2.1). Further,* *define the functions m and $\mathcal{B}m$ as in Proposition 2.1, and set $m(\,\cdot\,,r) := \mathcal{B}m(\,\cdot\,,r) := 0$ for $r \leq 0$. Then, there exists a bounded interval $I \subset \mathbb{R}$, that depends on the support of f but not on Ω, such that*

$$\frac{2}{\kappa_d^{(1)}} \mathcal{F}_d(\partial\Omega, m, x_0) = \int_{S^{d-1}} \int_I |x_0 - \Phi(y)|^{d-1} \partial_r^2 \mathcal{B}m\left(\Phi(y), \rho + |x_0 - \Phi(y)|\right) \ln|\rho| \,\mathrm{d}\rho \mathrm{d}s(y)$$
$$+ \int_{S^{d-1}} \int_I \frac{|x_0 - \Phi(y)|^{d-1}}{\rho + 2\,|x_0 - \Phi(y)|} \partial_r \mathcal{B}m\left(\Phi(y), \rho + |x_0 - \Phi(y)|\right) \mathrm{d}\rho \mathrm{d}s(y)\,. \quad (2.5)$$

Proof. For the case $d = 2$, the expression (2.5) has been derived in [33, Proposition 2.1(b)]. The general case $d \geq 2$ is shown in an analogous manner. □

Remark 2.3 (Alternative representations of $\mathcal{F}_d$ for domains of the form (1.4)). One can easily verify that for any domain Ω of the form (1.4), which is convex but possibly unbounded, the spherical parameterization $\Phi\colon S^{d-1} \setminus S \to \partial\Omega$ is defined almost everywhere in S^{d-1}. Since S is a set of surface measure zero, the integral identity (2.2) as well as the representations (2.3), (2.4), (2.5) of the universal back-projection formula $\mathcal{F}_d$ hold true also for these domains.

2.2. *Approximation by elliptic domains*

We already mentioned in the introduction that any domain of the form (1.4) can be approximated by elliptic domains (1.5). A precise statement of this fact is given by the following proposition.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^d$ be any domain of the form (1.4). Then, for every compact subset $K \subset \Omega$ and every point $x_0 \in K$, there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of elliptic domains $E_n \subset \mathbb{R}^d$ such that the following holds:*

(a) For all $n \in \mathbb{N}$, we have $K \subset E_n$.

(b) Let Φ_n and Φ be the spherical parametrizations of ∂E_n and $\partial\Omega$, respectively, around x_0. Then, there exists a set S of surface measure zero, such that

$$\lim_{n\to\infty} \Phi_n(y) = \Phi(y) \quad \textit{for all } \; y \in S^{d-1} \setminus S\,. \quad (2.6)$$

Proof. It is sufficient to verify the statement of the proposition for domains of the form (1.5)-(1.8). Then, the general case is obtained with a help of the appropriate affine transformation. Also, it can be easily checked that statement (b) follows from statement (a) after possibly passing to a subsequence. Therefore, we consider the proof of statement (b) only.

For any elliptic domain of the form (1.5), the Proposition holds trivially true. Consider now an elliptic paraboloid $\Omega = P$ as in (1.6). In [33], we demonstrated how to construct a sequence of elliptic domains in two spatial dimensions such that it satisfies condition (b). The bounding ellipsoids of such a sequence have the following properties: one focus is kept fixed, another focus moves along a line that goes through the first focus, and the axes of the ellipses are appropriately related. This construction can be generalized to an arbitrary dimension as we shall show next. For that purpose, define the sequence $(E_n)_{n\in\mathbb{N}}$ of elliptic domains by

$$E_n = \left\{ x \in \mathbb{R}^d \;\middle|\; \frac{2}{n} \sum_{i=1}^{d-1} \alpha_i x_i^2 \;+\; \frac{(x_d - T_n)^2}{n^2} < 1 \right\},$$

where $T_n := 1/4 + \sqrt{n^2 - n/2}$. For illustration purpose, the upper left picture in Figure 3 shows the cropped elliptic paraboloid $P \cap \{x \in \mathbb{R}^3 \mid x_3 \leq 2\}$ for $d = 3$ and $(\alpha_1, \alpha_2) = (1, 2)$, together with the cropped approximating elliptic domain $E_n \cap \{x \in \mathbb{R}^3 \mid x_3 \leq 2.5\}$ for $n = 4$.

Next, fix an arbitrary point $x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,d}) \in \Omega$. Let the set of directions $S_1^{d-1} \subset S^{d-1}$ be defined as

$$S_1^{d-1} := \left\{ y \in S^{d-1} \;\middle|\; \sum_{i=1}^{d-1} y_i^2 \neq 0 \right\},$$

and take an arbitrary direction $y = (y_1, \ldots y_d) \in S_1^{d-1}$ from this set. According to the definition of the spherical parametrization (2.1), there is a unique $t_n > 0$ such that $\Phi_n(y) = x_0 + t_n\, y \in \partial E_n$. From this condition, one derives the quadratic equation $a_n t_n^2 + b_n t_n + c_n = 0$ for t_n, where

$$\begin{aligned} a_n &= \sum_{i=1}^{d-1} \alpha_i y_i^2 + \frac{y_d^2}{2n}, \\ b_n &= 2 \sum_{i=1}^{d-1} \alpha_i x_{0,i} y_i + \frac{y_d(x_{0,d} - T_n)}{n}, \\ c_n &= \sum_{i=1}^{d-1} \alpha_i x_{0,i}^2 + \frac{(x_{0,d} - T_n)^2}{2n} - \frac{n}{2}. \end{aligned}$$

In the same way, there is a unique $t > 0$ such that $\Phi(y) = x_0 + t\, y \in \partial\Omega$ which gives the quadratic equation $at^2 + bt + c = 0$ for t, where

$$a = \sum_{i=1}^{d-1} \alpha_i y_i^2, \quad b = 2 \sum_{i=1}^{d-1} \alpha_i x_{0,i} y_i - y_d, \quad c = \sum_{i=1}^{d-1} \alpha_i x_{0,i}^2 - x_{0,d}.$$

For $y \in S_1^{d-1}$, one can show that the sequences a_n, b_n, c_n converge to a, b, c, respectively. Since the solution of the quadratic equation depends continuously on the coefficients of the equation this implies $t_n \to t$ as $n \to \infty$. This finally shows (2.6) with $S := S^{d-1} \setminus S_1^{d-1}$ being a set of surface measure zero.

For cylindric domains of the form (1.7) or (1.8), the approximation property (2.6) can be shown in a similar manner. Namely, for a parabolic cylinder $\Omega = C_{\text{par}}$ of the form (1.7), the sequence of elliptic domains E_n can be taken as

$$E_n = \left\{ x \in \mathbb{R}^d \;\middle|\; \frac{2}{n} \sum_{i=1}^{b} \alpha_i x_i^2 + \frac{1}{n^2} \sum_{i=b+1}^{d-1} x_i^2 + \frac{(x_d - T_n)^2}{n^2} < 1 \right\}.$$

One can show that (2.6) holds with $S = S^{d-1} \setminus S_2^{d-1}$, where $S_2^{d-1} := \{y \in S^{d-1} \mid \sum_{i=1}^{b} y_i^2 \neq 0\}$. Again for illustration purpose, the upper right picture in Figure 3 shows the cropped parabolic cylinder $C_{\text{par}} \cap \{x \in \mathbb{R}^3 \mid x_3 \le 9/4 \text{ and } x_2 \in [-3, 3]\}$ for $d = 3$ and $\alpha_1 = 1$, as well as the cropped approximating elliptic domain $E_n \cap \{x \in \mathbb{R}^3 \mid x_3 \le 3\}$ for $n = 4$.

Finally, for an elliptic cylinder C_{ell} of the form (1.8), we can verify (2.6) with

$$E_n = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{b} \alpha_i x_i^2 + \frac{1}{n} \sum_{i=b+1}^{d} x_i^2 < 1 \right\},$$

and $S := S^{d-1} \setminus S_2^{d-1}$. The lower left image in Figure 3 shows an cropped elliptic cylinder $C_{\text{ell}} \cap \{x \in \mathbb{R}^3 \mid x_3 \in [-1, 1]\}$ for $d = 3$, $b = 2$ and $(\alpha_1, \alpha_2) = (1, 2)$, together with the cropped approximating elliptic domain $E_n \cap \{x \in \mathbb{R}^3 \mid x_3 \in [-1.3, 1.3]\}$ for $n = 4$. For $b = 1$, the elliptic cylinder C_{ell} is bounded by two parallel lines. This is shown in the lower right image in Figure 3 which shows the cropped elliptic cylinder $C_{\text{ell}} \cap \{x \in \mathbb{R}^3 \mid x_2, x_3 \in [-1.5, 1.5]\}$ for $b = 1$ and $\alpha_1 = 1$ together with the approximating elliptic domain E_4. $\square$

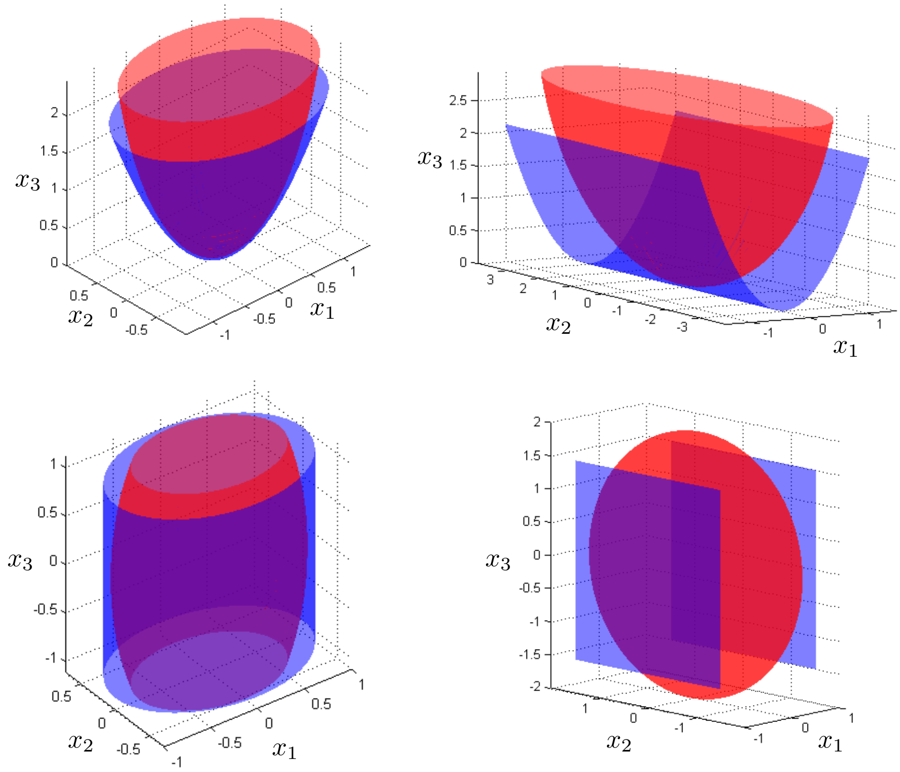

Figure 3: Illustration of the approximation of the quadrics of the forms (1.6), (1.7), and (1.8) in $\mathbb{R}^3$ (blue surfaces) by elliptic surfaces (red). Note that there arise two qualitatively different elliptic cylinders for the cases $b = 2$ (bottom left) and $b = 1$ (bottom right).

3. Proofs of the main results

Throughout this section, let $\Omega \subset \mathbb{R}^d$ denote a domain of the form (1.4), let $f \in C_c^\infty(\Omega)$, let $x_0 \in \Omega$ be a reconstruction point, and let $\Phi \colon S^{d-1} \setminus S \to \Omega$ be the spherical parameterization of $\partial\Omega$ around x_0, where S is a set of surface measure zero.

According to Proposition 2.4 applied with $K = \{x_0\} \cup \operatorname{supp}(f)$, we can choose a sequence of elliptic domains $(E_n)_{n\in\mathbb{N}}$ such that $x_0 \in E_n$, $\operatorname{supp}(f) \subset E_n$ and

$$\lim_{n\to\infty} \Phi_n(y) = \Phi(y) \quad \text{for every } y \in S^{d-1} \setminus S\,, \tag{3.1}$$

where $\Phi_n \colon S^{d-1} \to \partial E_n$ is the spherical parameterization of ∂E_n around x_0. We further define, for every $(x, r) \in \mathbb{R}^d \times (0, \infty)$,

$$\begin{aligned} m(x,r) &:= (\mathcal{M}f)\,(x,r)\,, \\ \mathcal{B}m(x,r) &:= \left(\mathcal{D}_r^{d-2} r^{d-2} m\right)(x,r)\,, \\ p(x,t) &:= (\mathcal{W}f)\,(x,t)\,. \end{aligned}$$

As in Proposition 2.2, we again set $m(\,\cdot\,, r) := \mathcal{B}m(\,\cdot\,, r) := 0$ for $r \le 0$.

3.1. Representation of radial derivatives of $\mathcal{B}m$

The proof of Theorem 1.2 is based on the application of the dominated convergence theorem. For that purpose we require estimates for the radial derivatives of $\mathcal{B}m$. Such estimates will be obtained using the following representations of $\partial_r \mathcal{B}m$ and $\partial_r^2 \mathcal{B}m$ in terms of derivatives of the spherical means m.

Lemma 3.1. *For every $d \geq 3$, there are constants $c_{d,i}^{(1)}, c_{d,i}^{(2)} \in \mathbb{R}$ such that*

$$\partial_r \mathcal{B}m = \sum_{i=0}^{d-1} c_{d,i}^{(1)} \, \frac{1}{r^{(d-1)-i}} \, \partial_r^i m \,, \tag{3.2}$$

$$\partial_r^2 \mathcal{B}m = \sum_{i=0}^{d} c_{d,i}^{(2)} \, \frac{1}{r^{d-i}} \, \partial_r^i m \,. \tag{3.3}$$

Proof. Let us first derive an expression for $\mathcal{B}m$ involving derivatives of m with respect to r. Since $\mathcal{D}_r$ is the differentiation operator with respect to r^2, one can use Leibniz's rule for the higher-order derivatives of a product of two functions to obtain the following representation of the function $\mathcal{B}m$:

$$\mathcal{B}m = \mathcal{D}_r^{d-2}\left(r^{d-2}\, m\right) = \sum_{k=0}^{d-2} \binom{d-2}{k} \mathcal{D}_r^{d-2-k}\left(r^{d-2}\right) \mathcal{D}_r^{k}\left(m\right). \tag{3.4}$$

Now one notes that $\mathcal{D}_r\left(r^l\right) = (l/2)\, r^{l-2}$ for any $l \neq 0$. Repeated application of this identity shows that there are constants $c_k^{(3)} \in \mathbb{R}$ such that the following hold:

- If $d \geq 3$ is odd, then

$$\mathcal{D}_r^k\left(r^{d-2}\right) = c_k^{(3)}\, r^{d-2-2k} \qquad \text{for all } k \geq 0\,. \tag{3.5}$$

- If $d \geq 2$ is even, then

$$\mathcal{D}_r^k\left(r^{d-2}\right) = \begin{cases} c_k^{(3)}\, r^{d-2-2k} & \text{for } 0 \leq k \leq (d-2)/2 \\ 0 & \text{for } k > (d-2)/2\,. \end{cases} \tag{3.6}$$

Next we use mathematical induction to show that there are constants $c_{k,i}^{(4)} \in \mathbb{R}$ such that for any $k \geq 1$ we have

$$\mathcal{D}_r^k(m) = \sum_{i=1}^{k} c_{k,i}^{(4)} \, \frac{1}{r^{2k-i}} \, \partial_r^i m \,. \tag{3.7}$$

Indeed, for $k = 1$, we have $\mathcal{D}_r(m) = (2r)^{-1}\partial_r m$ which is of the form (3.7). Now suppose that the representation (3.7) holds true for some $k \geq 1$. Then,

$$\begin{aligned}
\mathcal{D}_r^{k+1}(m) &= \frac{1}{2r}\partial_r \sum_{i=1}^{k} c_{k,i}^{(4)} \, \frac{1}{r^{2k-i}} \, \partial_r^i m \\
&= \sum_{i=1}^{k} \frac{c_{k,i}^{(4)}}{2} \left(\frac{-2k+i}{r^{2k-i+2}} \, \partial_r^i m + \frac{1}{r^{2k-i+1}} \, \partial_r^{i+1}\, m \right) \\
&= \frac{c_{k,1}^{(4)}}{2} \, \frac{-2k+1}{r^{2(k+1)-1}} \, \partial_r m \; + \; \frac{c_{k,k}^{(4)}}{2} \, \frac{1}{r^{k+1}} \, \partial_r^{k+1} m \\
&\qquad + \sum_{i=2}^{k} \left(\frac{c_{k,i}^{(4)}(-2k+i)}{2\, r^{2(k+1)-i}} \, \partial_r^i m + \frac{c_{k,i-1}^{(4)}}{2\, r^{2(k+1)-i}} \, \partial_r^i m \right).
\end{aligned} \tag{3.8}$$

The last expression has the form (3.7), and therefore (3.7) holds for all $k \geq 1$.

Now, consider a non-zero term $\mathcal{D}_r^{d-2-k}\left(r^{d-2}\right)\mathcal{D}_r^k(m)$ in the sum (3.4). Using (3.5)-(3.7), for $k \geq 1$, this term can be written as follows:

$$\begin{aligned}\mathcal{D}_r^{d-2-k}\left(r^{d-2}\right)\mathcal{D}_r^k(m) &= c_{d-2-k}^{(3)}\, r^{d-2-2(d-2-k)} \sum_{i=1}^{k} c_{k,i}^{(4)} \frac{1}{r^{2k-i}}\, \partial_r^i m \\ &= \sum_{i=1}^{k} c_{d-2-k}^{(3)} c_{k,i}^{(4)} \frac{1}{r^{(d-2)-i}}\, \partial_r^i m. \end{aligned} \tag{3.9}$$

For $k = 0$, we have

$$\mathcal{D}_r^{d-2}\left(r^{d-2}\right)\, m = \begin{cases} 0, & \text{if } d \geq 4 \text{ is even}\,, \\ c_{d-2}^{(3)} \frac{1}{r^{d-2}}\, m, & \text{if } d \geq 3 \text{ is odd}\,. \end{cases} \tag{3.10}$$

From (3.9) and (3.10) we conclude that

$$\mathcal{B}m = \mathcal{D}_r^{d-2}\left(r^{d-2}\, m\right) = \sum_{i=0}^{d-2} c_{d,i}^{(5)} \frac{1}{r^{(d-2)-i}}\, \partial_r^i m\,, \tag{3.11}$$

for certain constants $c_{d,i}^{(5)} \in \mathbb{R}$. Expressions (3.2) and (3.3) for the radial derivatives of $\mathcal{B}m$ follow from (3.11) by applying the radial derivative ∂_r and performing similar algebraic manipulations as in (3.8). □

3.2. Proof of Theorem 1.2

3.2.1. Odd dimension

Let $d \geq 3$ be an odd integer. According to Theorem 1.1, the formula $\mathcal{F}_d$ exactly recovers f from data (1.2) for every elliptic domain E_n. Then, with the representation (2.4), we have

$$f\left(x_0\right) = -\kappa_d^{(1)} \int_{S^{d-1}} |x_0 - \Phi_n(y)|^{d-1} \partial_r\, \mathcal{B}m\left(\Phi_n(y), |x_0 - \Phi_n(y)|\right) \mathrm{d}s(y)\,. \tag{3.12}$$

Since the formula (3.12) holds true for all $n \in \mathbb{N}$, we have

$$f\left(x_0\right) = -\kappa_d^{(1)} \lim_{n\to\infty} \int_{S^{d-1}} I_n(y) \mathrm{d}s(y)\,, \text{ where} \tag{3.13}$$

$$I_n(y) := |x_0 - \Phi_n(y)|^{d-1}\, \partial_r \mathcal{B}m\left(\Phi_n(y), |x_0 - \Phi_n(y)|\right)\,. \tag{3.14}$$

In the next Lemma, we show that the integrands I_n are uniformly bounded by an integrable function which allows to apply the dominated convergence theorem to (3.13).

Lemma 3.2. *For any $f \in C_c^\infty(\Omega)$, there is a constant $c_1 > 0$ such that for all $n \in \mathbb{N}$ and all $y \in S^{d-1}$, we have $|I_n(y)| \leq c_1$.*

Proof. First notice that $I_n(y) = (r^{d-1}\partial_r\, \mathcal{B}m)\left(\Phi_n(y), |x_0 - \Phi_n(y)|\right)$. According to Lemma 3.1 there are constants $c_{d,i}^{(1)}$ such that

$$r^{d-1}\partial_r\, \mathcal{B}m = \sum_{i=0}^{d-1} c_{d,i}^{(1)}\, r^i\, \partial_r^i m. \tag{3.15}$$

Next note that the spherical means of any smooth function f with compact support together with its its radial derivatives satisfy $\left|\partial_r^i m(x,r)\right| \leq C_i \min\left\{r^{-(d-1)}, 1\right\}$ for all $(x,r) \in \mathbb{R}^d \times$

$(0,\infty)$, where C_i are some positive constants that depend only on f. Thus, in the view of (3.15), we can estimate

$$\left|r^{d-1}\partial_r \mathcal{B}m(x,r)\right| \leq \sum_{i=0}^{d-1} \left|c_{d,i}^{(1)}\right| C_i r^i \min\left\{r^{-(d-1)}, 1\right\} =: b(r)$$

for all $(x,r) \in \mathbb{R}^d \times (0,\infty)$. It can be easily verified that $r \mapsto b(r)$ is bounded on $(0,\infty)$. Consequently, the functions $y \mapsto I_n(y) = (r^{d-1}\partial_r \mathcal{B}m)(\Phi_n(y), |x_0 - \Phi_n(y)|)$ are uniformly bounded which concludes the proof of the Lemma. □

Now, we continue the proof of Theorem 1.2. According to the above Lemma, we can apply the dominated convergence theorem to (3.13). Since the function $\partial_r\mathcal{B}m(x,r)$ depends continuously on both of its arguments, and Φ_n converges pointwise to Φ almost everywhere on S^{d-1}, we get

$$\begin{aligned} f(x_0) &= -\kappa_d^{(1)} \int_{S^{d-1}} \lim_{n\to\infty} I_n(y) \mathrm{d}s(y) \\ &= -\kappa_d^{(1)} \int_{S^{d-1}} |x_0 - \Phi(y)|^{d-1} \partial_r \mathcal{B}m\left(\Phi(y), |x_0 - \Phi(y)|\right) \mathrm{d}s(y)\,. \end{aligned}$$

Together with Remark 2.3 this implies identity (1.13) for the case of odd dimension.

3.2.2. Even dimension

We next consider the case of even spatial dimension. For $d = 2$, Theorem 1.2 has been proven in [33]. Therefore, we assume that $d \geq 4$ the following.

According to Theorem 1.1 and Proposition 2.2 we have

$$\begin{aligned} \frac{2}{\kappa_d^{(1)}} f(x_0) &= \int_{S^{d-1}} \int_I |x_0 - \Phi_n(y)|^{d-1}\, \partial_r^2 \mathcal{B}m\left(\Phi_n(y), \rho + |x_0 - \Phi_n(y)|\right) \ln|\rho|\, \mathrm{d}\rho\, \mathrm{d}s(y) \\ &+ \int_{S^{d-1}} \int_I \frac{|x_0 - \Phi_n(y)|^{d-1}}{\rho + 2\,|x_0 - \Phi_n(y)|}\, \partial_r \mathcal{B}m\left(\Phi_n(y), \rho + |x_0 - \Phi_n(y)|\right) \mathrm{d}\rho\, \mathrm{d}s(y)\,. \end{aligned} \tag{3.16}$$

Let us denote the integrands in the above expression by $I_{1,n} \colon S^{d-1} \times I \to \mathbb{R}$ and $I_{2,n} \colon S^{d-1} \times I \to \mathbb{R}$, that is

$$\begin{aligned} I_{1,n}(y,\rho) &:= |x_0 - \Phi_n(y)|^{d-1}\, \partial_r^2 \mathcal{B}m\left(\Phi_n(y), \rho + |x_0 - \Phi_n(y)|\right) \ln|\rho|\,, \\ I_{2,n}(y,\rho) &:= \frac{|x_0 - \Phi_n(y)|^{d-1}}{\rho + 2\,|x_0 - \Phi_n(y)|}\, \partial_r \mathcal{B}m\left(\Phi_n(y), \rho + |x_0 - \Phi_n(y)|\right)\,. \end{aligned}$$

As the identity (3.16) holds for all $n \in \mathbb{N}$ we have

$$\frac{2}{\kappa_d^{(1)}} f(x_0) = \lim_{n\to\infty} \int_{S^{d-1}} \int_I I_{1,n}(y,\rho)\, \mathrm{d}\rho\, \mathrm{d}s(y) + \lim_{n\to\infty} \int_{S^{d-1}} \int_I I_{2,n}(y,\rho)\, \mathrm{d}\rho\, \mathrm{d}s(y)\,. \tag{3.17}$$

For the application of the dominated convergence theorem to (3.17), we next show that the integrands $I_{1,n}$ and $I_{2,n}$ are uniformly bounded by integrable functions.

Lemma 3.3. *For any $f \in C_c^\infty(\Omega)$, there are constants $c_1 > 0$, $c_2 > 0$ such that, for all $n \in \mathbb{N}$ and all $(y,\rho) \in S^{d-1} \times I$, the following hold:*

$$|I_{1,n}(y,\rho)| \leq c_1\, |\ln|\rho||\,, \tag{3.18}$$

$$|I_{2,n}(y,\rho)| \leq c_2\,. \tag{3.19}$$

Proof. Define the function $\widehat{m}_1(x,r,\rho) := r^{d-1}\partial_r^2\,\mathcal{B}m(x,\rho+r)$. Then,

$$I_{1,n}(y,\rho) = \widehat{m}_1(\Phi_n(y), |x_0-\Phi_n(y)|\,,\rho)\,\ln|\rho|\,. \tag{3.20}$$

According to the choice of $(E_n)_{n\in\mathbb{N}}$, there exists $r_*>0$ only depending on the function f, such that $m(x,r)=\mathcal{B}m(x,r)=0$ for all $r<r_*$ and $x\in\bigcup_{n\in\mathbb{N}}E_n$. Thus, in view of (3.20), for proving the bound (3.18), it is sufficient to show that there exists a constant $c_1>0$ such that $|\widehat{m}_1(x,r,\rho)|\le c_1$ for all $x\in\bigcup_{n\in\mathbb{N}}E_n$ and $(r,\rho)\in T_* := \{(r,\rho)\in(0,\infty)\times I\ \mid r+\rho\ge r_*\}$.

With the representation (3.3) for the function $\partial_r^2\,\mathcal{B}m$, and using the estimates $|\partial_r^i m(x,r)|\le C_i\min\{r^{-(d-1)},1\}$ we obtain

$$|\widehat{m}_1(x,r,\rho)| \le \sum_{i=0}^{d}\left|c_{d,i}^{(2)}\right| C_i\frac{1}{(\rho+r)^{d-i}}r^{d-1}\,\min\left\{(\rho+r)^{-(d-1)},1\right\}$$

for $x\in\bigcup_{n\in\mathbb{N}}E_n$ and $(r,\rho)\in T_*$. The functions $r^{d-1}\min\left\{(\rho+r)^{-(d-1)},1\right\}$ and $(\rho+r)^{-(d-i)}$, for $i\in\{0,1,\dots,d\}$, are bounded for $(r,\rho)\in T_{r_*}$ which proves the first estimate (3.18). The second estimate (3.19) is shown in a similar manner. $\square$

Lemma 3.3 allows application of the dominated convergence theorem to (3.17). As in the case of odd spatial dimension d, this yields

$$\begin{aligned}\frac{2}{\kappa_d^{(1)}}f(x_0) &= \int_{S^{d-1}}\int_I |x_0-\Phi(y)|^{d-1}\,\partial_r^2\,\mathcal{B}m\left(\Phi(y),\rho+|x_0-\Phi(y)|\right)\ln|\rho|\,\mathrm{d}\rho\,\mathrm{d}s(y)\\ &\quad+\int_{S^{d-1}}\int_I\frac{|x_0-\Phi(y)|^{d-1}}{\rho+2\,|x_0-\Phi(y)|}\,\partial_r\mathcal{B}m\left(\Phi(y),\rho+|x_0-\Phi(y)|\right)\,\mathrm{d}\rho\,\mathrm{d}s(y)\,.\end{aligned}$$

Together with Remark 2.3 this proves the statement of Theorem 1.2 for even $d\ge4$.

3.3. *Proof of Theorem 1.3*

3.3.1. *Odd dimension*

We first consider the case of odd dimension $d\ge3$. Identity (1.13) for the spherical means m in odd spatial dimension states

$$f(x_0) = \frac{(-1)^{(d-3)/2}\omega_{d-1}}{4\pi^{d-1}}\int_{\partial\Omega}\frac{\langle\nu_x,x_0-x\rangle}{|x_0-x|}\left(\partial_r\mathcal{D}_r^{d-2}r^{d-2}m\right)(x,|x_0-x|)\,\mathrm{d}s(x)\,. \tag{3.21}$$

It is known (see, for example, [34, p. 682]), that in the case of odd spatial dimension the solution p of the initial value problem (1.1) can be expressed through the spherical means $m=\mathcal{M}f$ as

$$p(x,r) = \frac{\omega_{d-1}}{4\pi^{(d-1)/2}}\left(\partial_r\mathcal{D}_r^{(d-3)/2}r^{d-2}m\right)(x,r).$$

Application of $\mathcal{D}_r^{(d-3)/2}\,r^{-1}$ yields

$$\left(\mathcal{D}_r^{(d-3)/2}r^{-1}p\right)(x,r) = \frac{\omega_{d-1}}{2\pi^{(d-1)/2}}\,\left(\mathcal{D}_r^{d-2}r^{d-2}m\right)(x,r). \tag{3.22}$$

Equations (3.21) and (3.22) yield

$$f(x_0) = \frac{(-1)^{(d-3)/2}}{2\pi^{(d-1)/2}}\int_{\partial\Omega}\frac{\langle\nu_x,x_0-x\rangle}{|x_0-x|}\left(\partial_t\mathcal{D}_t^{(d-3)/2}t^{-1}p\right)(x,|x_0-x|)\,\mathrm{d}s(x)\,. \tag{3.23}$$

which is the desired identity for the odd dimensional case.

3.3.2. Even dimension

In the case of even spatial dimension $d \geq 2$, the inversion formula (1.14) is given by

$$f(x_0) = \frac{(-1)^{(d-2)/2}}{\pi^{d/2}} \int_{\partial\Omega} \langle \nu_x, x_0 - x \rangle \int_{|x_0-x|}^{\infty} \frac{\left(\partial_t \mathcal{D}_t^{(d-2)/2} t^{-1} p\right)(x,t)}{\sqrt{t^2 - |x_0-x|^2}} \, \mathrm{d}t \, \mathrm{d}s(x) . \tag{3.24}$$

As in the proof of the odd dimensional case, we employ the inversion formula (1.13) for the spherical means which states

$$f(x_0) = \frac{(-1)^{(d-2)/2}\omega_{d-1}}{2\pi^d} \int_{\partial\Omega} \langle \nu_x, x_0 - x \rangle \int_0^{\infty} \frac{\left(\partial_r \mathcal{D}_r^{d-2} r^{d-2} m\right)(x,r)}{r^2 - |x_0-x|^2} \, \mathrm{d}r \, \mathrm{d}s(x) . \tag{3.25}$$

In the case of even dimension, the solution p of the initial value problem (1.1) and the spherical means $m = \mathcal{M}f$ satisfy the following relation (see [6, p. 226])

$$\frac{1}{2\pi^{d/2}} \int_{|x_0-x|}^{\infty} \left(\mathcal{D}_t^{(d-2)/2} \frac{\chi\left\{t^2 - |x_0-x|^2 > 0\right\}}{\sqrt{t^2 - |x_0-x|^2}} \right) p(x,t) \, \mathrm{d}t = \\ \frac{(-1)^{(d-2)/2}\omega_{d-1}}{4\pi^d} \int_0^{\infty} \frac{\left(r \mathcal{D}_r^{d-2} r^{d-2} m\right)(x,r)}{r^2 - |x_0-x|^2} \, \mathrm{d}r ,$$

where $\chi\{t^2 - |x|^2 > 0\}$ is the characteristic function of $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} \mid t^2 - |x|^2 > 0\}$ and the expression on the left has to be understood in the distributional sense. Using integration by parts in the integrals on the left and on the right hand side yields

$$\frac{(-1)^{(d-2)/2}}{\pi^{d/2}} \int_{|x_0-x|}^{\infty} \sqrt{t^2 - |x_0-x|^2} \left(\partial_t \mathcal{D}_t^{(d-2)/2} t^{-1} p\right)(x,t) \, \mathrm{d}t = \\ \frac{(-1)^{(d-2)/2}\omega_{d-1}}{4\pi^d} \int_0^{\infty} \ln\left|r^2 - |x_0-x|^2\right| \left(\partial_r \mathcal{D}_r^{d-2} r^{d-2} m\right)(x,r) \, \mathrm{d}r. \tag{3.26}$$

Applying the gradient with respect to x_0 on the both sides of (3.26) further shows

$$\frac{(-1)^{(d-2)/2}}{\pi^{d/2}} (x_0 - x) \int_{|x_0-x|}^{\infty} \frac{\left(\partial_t \mathcal{D}_t^{(d-2)/2} t^{-1} p\right)(x,t)}{\sqrt{t^2 - |x_0-x|^2}} \, \mathrm{d}t = \\ \frac{(-1)^{(d-2)/2}\omega_{d-1}}{2\pi^d} (x_0 - x) \int_0^{\infty} \frac{\left(\partial_r \mathcal{D}_r^{d-2} r^{d-2} m\right)(x,r)}{r^2 - |x_0-x|^2} \, \mathrm{d}r. \tag{3.27}$$

The relationship (3.27) together with the inversion formula (3.25) for the spherical means m implies the desired inversion formula (3.24) for the wave data p.

4. Discussion

In this paper we studied the problems of reconstructing a function from its spherical means or the solution of the standard free-space wave equation on quadrics bounding domains of the form (1.4). We showed that the universal back-projection formula, originally introduced for three spatial dimensions in the context of photoacoustic tomography in [1], provides exact reconstruction for these quadrics including parabolas as well as parabolic and elliptic cylinders. Note that the universal back-projection has previously been shown to provide an exact reconstruction on spherical and cylindrical domains in three spatial dimensions in [1], spherical domains in any dimension in [2], elliptical domains in three dimensions in [3] and later for elliptical domains in arbitrary dimension in [6]. The results of the present paper have been derived by using the formulas of [6] for elliptical domains and application of the dominated convergence theorem to receive the corresponding formulas for domains (1.4), which can be approximated by elliptic domains.

Formulas different from the universal back-projection formulas that also exactly recover a function from spherical means on elliptic domains have been derived in [30, 31, 32, 35]. It would be interesting to study whether some of the formulas of [30, 31, 32, 35] for elliptical domains work for domains (1.4) as well. It would also be interesting to clarify the relationships between the various formulas for elliptical or more general quadric domains. Note that the formula of [31, 35] generalize one of the formulas of [26, 27] from spherical to elliptical center set. For the special case of spherical domains such relations have been instigated in [9, 5]. In [9] it has been shown that for a spherical observation surface in two and three spatial dimensions, the universal back-projection formula can be derived from the inversion formulas of [26, 27]. In [28] a class of inversion formulas has been obtained that includes the formulas of [26, 27] as well as the universal back-projection formulas.

Another open question is whether the universal back-projection formula provides exact reconstruction from spherical means in situations different from the ones considered in the present paper. For example, recently in [36] the formula of [32] has been shown to provide exact reconstruction in the case that the observation surface is a zero set of certain polynomials (named oscillatory sets in [36]), and that the function to be recovered is supported in a certain associated domain (named hyperbolic cavity in [36]). It is therefore tempting to analyze the behaviour of the universal back-projection in such a situation as well. After one of the authors presented the results of [6] at the 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications 2014 in Madrid such a study has also been suggested by Prof. Mark Agranovsky.

Finally, note that in the practical implementation the observation surface has to be replaced by a bounded subset of the boundary $\partial\Omega$. This limited data problem is well known to introduce artifacts when using explicit inversion formulas, see [37, 38, 39, 40]. Limited data artifacts using the universal back-projection formula on an elliptic or parabolic observation surface have been recently theoretically analyzed in [5]; compare also [41, 42, 18]. Further theoretical aspects of spherical means that we have not touched in this paper can be found, for example, in [43, 44, 45, 46, 47, 48, 49] and the references therein.

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