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Exact Reconstruction Formula for the Spherical Mean Radon Transform on Ellipsoids

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Abstract

Many modern imaging and remote sensing applications require reconstructing a function from spherical averages (mean values). Examples include photoacoustic tomography, ultrasound imaging or SONAR. Several formulas of the back-projection type for recovering a function in n spatial dimensions from mean values over spheres centered on a sphere have been derived in [D. Finch, S. K. Patch, and Rakesh. Determining a function from its mean values over a family of spheres. SIAM J. Math. Anal., 35(5):1213-1240, 2004] for odd spatial dimension and in [D. Finch, M. Haltmeier, and Rakesh, SIAM J. Appl. Math. 68(2), pp. 392-412, 2007] for even spatial dimension. In this paper we generalize some of these formulas to the case where the centers of integration lie on the boundary of an arbitrary ellipsoid. For the special cases $n = 2$ and $n = 3$ our results have recently been established in [Y. Salman, arXiv:1208.5739 (math.AP), 2012].

Keywords. Spherical means, Radon transform, Inversion, Reconstruction formula, back-projection, Ellipsoids.

AMS classification numbers. 35L05, 44A12, 65R32, 92C55.

1 Introduction

The spherical mean Radon transform $\mathcal{M}\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n \times (0,\infty))$ maps a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ to the spherical mean values

$$(\mathcal{M}f)(z,r) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(z + r\omega)\,\mathrm{d}\omega\,, \qquad \text{for } (z,r) \in \mathbb{R}^n \times (0,\infty)\,. \tag{1.1}$$

Here $S^{n-1} = \{\omega \in \mathbb{R}^n \colon |\omega| = 1\}$ is the $n-1$ dimensional unit sphere, $\omega_{n-1} := |S^{n-1}|$ its total surface measure, and $\mathrm{d}\omega$ the standard surface measure on S^{n-1}. The value $(\mathcal{M}f)(z,r)$ is the average of f over a sphere with center $z \in \mathbb{R}^n$ and radius $r > 0$.

In this paper we study the problem of recovering a function f supported in an ellipsoid $E \subset \mathbb{R}^n$ from the spherical mean values $(\mathcal{M}f)(z, r)$ with centers restricted to the boundary of the ellipsoid. Recovering a function from spherical mean values with restricted centers is essential for many imaging and remote sensing applications, such as photoacoustic and thermoacoustic tomography (see [7, 15, 22, 39]), SONAR (see [5, 36]), or ultrasound tomography (see [30, 31]).

In [13, 14] several explicit reconstruction formulas for the spherical mean Radon transform have been derived for recovering a function supported in an n-dimensional ball from its spherical mean values centered on the boundary sphere. The proofs of [13, 14] are based on integral identities established directly for $n = 2$ and $n = 3$. The higher dimensional cases have been reduced to the two and three dimensional cases by expanding the function to be recovered in a series of spherical harmonics. In [37] one of the formulas of [13, 14] as well as the methods of proofs have been extended to elliptical domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. In the present paper we generalize these formulas to elliptical domains in arbitrary dimension. Opposed to [13, 14] the proofs for the higher dimensional cases are not based on a spherical harmonic expansion, which may be difficult to generalize to elliptical domains. Instead we present direct proofs that generalize the proofs of [13, 14] from spherical center sets in two and three dimensions to elliptical center sets in arbitrary dimensions.

1.1 Notation

Throughout this paper, let $a_1, \dots, a_n > 0$ be given numbers, let $A := \operatorname{diag}(a_1, \dots, a_n)$ denote the diagonal matrix with diagonal entries a_i, and let

$$E := \left\{x \in \mathbb{R}^n : \left|A^{-1}x\right| < 1\right\} = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} < 1\right\} \tag{1.2}$$

denote the corresponding solid ellipsoid with semi-principal axes $a_1, \dots, a_n$. In (1.2) and below $|\cdot|$ is the Euclidian norm on $\mathbb{R}^n$; the corresponding inner product will be denoted by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Any point on the boundary of E will be written in the form $A\sigma \in \partial E$ with $\sigma \in S^{n-1}$.

We denote by $C^\infty(\mathbb{R}^n)$ the set of all smooth (that is infinitely times differentiable) functions and by $C_0^\infty(\bar{E})$ the subset of all smooth functions with support contained in $\bar{E}$. The spherical means $(\mathcal{M}f)(z, r)$ of a function $f \in C^\infty(\mathbb{R}^n)$ are defined by (1.1). We are in particular interested in the case where $f \in C_0^\infty(\bar{E})$ and where the centers of integration are restricted to ∂E. Here and below we use $C_0^\infty(\bar{E})$ to denote the set of all smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp}(f) \subset \bar{E}$. Further we denote by

$$\Delta_{Ax} := \sum_{i=1}^{n} \frac{1}{a_i^2} \frac{\partial^2}{\partial x_i^2}$$

the Laplacian with respect to the variable Ax and by $\mathcal{D}_r := \frac{1}{2r}\frac{\partial}{\partial r}$ the differentiation operator with respect to r^2. Occasionally we make use of the of the Hilbert transform $\mathcal{H}_s$ defined as the convolution with the distribution $\mathrm{P.V.}(1/\pi s)$. We will also frequently write r^k for the multiplication operator that maps a function $(z,r) \mapsto g(z,r)$ to the function $(z,r) \mapsto r^k g(z,r)$.

An important role in our analysis plays the function $G_n \colon \mathbb{R}^{n\times n} \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ defined by

$$G_n(x,y) = \begin{cases} \frac{1}{2\pi} \log|x-y| & \text{for } n=2 \\ \frac{|x-y|^{2-n}}{\omega_{n-1}(2-n)} & \text{for } n>2\,, \end{cases}$$

which is the fundamental solution of n-dimensional Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. By definition, the fundamental solution G_n is a solution of $\Delta_x G_n(x,y) = \delta_n(x-y)$, with δ_n denoting the n-dimensional delta distribution.

1.2 Main results

As the main result of this paper we derive explicit reconstruction formulas for recovering a smooth function supported inside the ellipsoid $\bar{E}$ from its spherical mean values with centers on the boundary ∂E.

In even dimension our main result reads as follows:

Theorem 1.1 (Reconstruction in even dimension)**.**
Let $n \geq 2$ be even, let E be the solid ellipsoid defined by (1.2), and define the constant $c_n := (-1)^{(n-2)/2}\omega_{n-2}\pi(n-2)!\,2^{2-n}$.

Then, for every $f \in C_0^\infty(\bar{E})$ and every $x \in E$, we have

$$f(x) = \frac{\det(A)}{c_n} \Delta_{Ax} \int_{S^{n-1}} \int_0^\infty \left(r\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M}f\right)(A\sigma, r) \\ \times \log\left||x-A\sigma|^2 - r^2\right| \mathrm{d}r \mathrm{d}\sigma\,. \tag{1.3}$$

Here ∂_r denotes differentiation with respect to r, $\mathcal{D}_r = (2r)^{-1}\partial_r$ differentiation with respect to r^2, and $\Delta_{Ax} = \sum_{i=1}^n \frac{1}{a_i^2}\partial_{x_i}^2$ the Laplacian with respect to Ax.

Proof. See Section 3. □

In odd dimensions we have the following corresponding result:

Theorem 1.2 (Reconstruction in odd dimension)**.**
Let $n \geq 3$ be odd, let E be the solid ellipsoid defined by (1.2), and define the constant $c_n := (-1)^{(n-1)/2}\omega_{n-2}(n-2)!\,2^{n-3}$.

Then, for every $f \in C_0^\infty(\bar{E})$ and every $x \in E$, we have

$$f(x) = \frac{\det(A)}{c_n} \Delta_{Ax} \int_{S^{n-1}} \left(r\mathcal{D}_r^{n-3} r^{n-2} \mathcal{M}f\right)\left(A\sigma, |x-A\sigma|^2\right) \mathrm{d}\sigma\,. \tag{1.4}$$

Here ∂_r, $\mathcal{D}_r$ and Δ_{Ax} are as in Theorem 1.1.

Proof. See Section 4. □

If all semi-principal axis $a_1, \dots, a_n$ coincide, then the ellipsoid E is obviously an n-dimensional ball. In such a case, the reconstruction formulas of Theorem 1.1 and 1.2 have been first established in [13] for even dimensions and in [14] for odd dimensions. For the special cases of elliptical domains in two and three dimension, the formulas of Theorem 1.1 and 1.2 have been recently established in [37]. Different reconstruction formulas of the back-projection type for spherical means on ellipsoids have recently been obtained for two and three spatial dimensions in [4, 17, 29] and for arbitrary dimension in [18, 32].

1.3 Outline

The remainder of this paper is mainly devoted to the proofs of Theorems 1.1 and 1.2. In the following Section 2 we derive some required auxiliary lemmas. The proof of Theorem 1.1 will be given in Section 3 and the proof of Theorem 1.2 will be given in Section 4. The paper concludes with a short discussion in Section 5.

2 Auxiliary results

Throughout the following, A denotes the diagonal matrix with positive diagonal entries $a_1, \dots, a_n > 0$, and E denotes the corresponding solid ellipsoid defined by (1.2).

Definition 2.1 (Reconstruction integral)**.**
For $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $f \in C_0^\infty(\mathbb{R}^n)$ we define

$$(\mathcal{B}_\Phi \mathcal{M} f)(x) := \int_{S^{n-1}} \int_0^\infty r \mathcal{D}_r^{n-2} r^{n-2} (\mathcal{M} f)(A\sigma, r) \\ \times \Phi\left(|x - A\sigma|^2 - r^2\right) \mathrm{d}r \mathrm{d}\sigma \,. \tag{2.1}$$

In the even dimensional case the reconstruction integral will be applied with $\Phi(s) = \log|s|$ whereas in odd dimensions we use (2.1) with $\Phi(s) = \chi\{s > 0\}$. In both cases we will show that $\Delta_{Ax} \mathcal{B}_\Phi \mathcal{M} f$ is a constant multiple of f, which yields the reconstruction formulas of Theorems 1.1 and 1.2 (see Sections 3 and 4).

Lemma 2.2 (Basic integral identity)**.**
Let $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R})$ be locally integrable and let $\Phi^{(n-2)}$ denote its $n-2$ fold distributional derivative. Then, for every $x \in \mathbb{R}^n$,

$$(\mathcal{B}_\Phi \mathcal{M} f)(x) = \frac{\omega_{n-2}}{\omega_{n-1}} \int_{\mathbb{R}^n} f(y) \int_{-1}^1 \left(1 - s^2\right)^{(n-3)/2} \\ \times \Phi^{(n-2)}\left(2\,|Ax - Ay| \left(\frac{|x|^2 - |y|^2}{2\,|A(x-y)|} - s\right)\right) \mathrm{d}s \mathrm{d}r \,. \tag{2.2}$$

Proof. Since $\mathcal{D}_r = (2r)^{-1}\partial_r$, we have $r\mathcal{D}_r^{n-2} = (-1)^{n-2}(\mathcal{D}_r^{n-2})^* r$ with $(\mathcal{D}_r^{n-2})^*$ denoting the formal L^2-adjoint of $\mathcal{D}_r^{n-2}$. Integration by parts and one application of

Fubini's theorem therefore yields

$$
\begin{aligned}
(\mathcal{B}_\Phi \mathcal{M} f)(x) &= \int_{S^{n-1}} \int_0^\infty \left(r \mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f \right) (A\sigma, r)\, \Phi\left(|x - A\sigma|^2 - r^2 \right) \mathrm{d}r \mathrm{d}\sigma \\
&= (-1)^{n-2} \int_{S^{n-1}} \int_0^\infty \left(\mathcal{D}_r^{n-2} \right)^* \left(r^{n-1} \mathcal{M} f \right) (A\sigma, r)\, \Phi\left(|x - A\sigma|^2 - r^2 \right) \mathrm{d}r \mathrm{d}\sigma \\
&= (-1)^{n-2} \int_{S^{n-1}} \int_0^\infty r^{n-1} \left(\mathcal{M} f \right) (A\sigma, r)\, \mathcal{D}_r^{n-2} \Phi\left(|x - A\sigma|^2 - r^2 \right) \mathrm{d}r \mathrm{d}\sigma \\
&= \int_0^\infty r^{n-1} \left(\mathcal{M} f \right) (A\sigma, r) \int_{S^{n-1}} \Phi^{(n-2)}\left(|x - A\sigma|^2 - r^2 \right) \mathrm{d}\sigma \mathrm{d}r \,.
\end{aligned}
$$

The last identity, the definition of $\left(\mathcal{M} f \right)(A\sigma, r)$, and the use of spherical coordinates $(r, \omega) \mapsto A\sigma + r\omega$ around the center $A\sigma$ show

$$
\left(\mathcal{B}_\Phi \mathcal{M} f \right)(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} f(y) \int_{S^{n-1}} \Phi^{(n-2)}\left(|x - A\sigma|^2 - |y - A\sigma|^2 \right) \mathrm{d}\sigma \mathrm{d}y \tag{2.3}
$$

Next we write the argument of $\Phi^{(n-2)}$ as

$$
\begin{aligned}
|x - A\sigma|^2 - |y - A\sigma|^2 &= |x|^2 - |y|^2 - 2\langle A\sigma, x - y \rangle \\
&= |x|^2 - |y|^2 - 2\langle \sigma, Ax - Ay \rangle \,.
\end{aligned}
$$

By the Funk-Hecke formula $\int_{S^{n-1}} h\left(\langle \sigma, v \rangle \right) \mathrm{d}\sigma = \omega_{n-2} \int_{-1}^1 \left(1 - s^2 \right)^{(n-3)/2} h\left(|v|\, s \right) \mathrm{d}s$ (see, for example, [27, 28]), the inner integral in (2.3) can be rewritten as

$$
\begin{aligned}
&\int_{S^{n-1}} \Phi^{(n-2)}\left(|x - A\sigma|^2 - |y - A\sigma|^2 \right) \mathrm{d}\sigma \\
&\quad = \int_{S^{n-1}} \Phi^{(n-2)}\Big(|x|^2 - |y|^2 - 2\langle \sigma, Ax - Ay \rangle \Big) \mathrm{d}\sigma \\
&\quad = \omega_{n-2} \int_{-1}^1 \left(1 - s^2 \right)^{(n-3)/2} \Phi^{(n-2)}\Big(|x|^2 - |y|^2 - 2\, |Ax - Ay|\, s \Big) \mathrm{d}s \\
&\quad = \omega_{n-2} \int_{-1}^1 \left(1 - s^2 \right)^{(n-3)/2} \Phi^{(n-2)}\left(2\, |Ax - Ay| \left(\frac{|x|^2 - |y|^2}{2\, |A(x - y)|} - s \right) \right) \mathrm{d}s \,.
\end{aligned}
$$

Inserting this for the inner integral in (2.3) yields (2.2). □

In the following sections we will show that for special choices of Φ the kernel in (2.3) is a constant multiple of the fundamental solution of the Laplace equation. For that purpose we will require that $(|x|^2 - |y|^2)/(2|A(x - y)|) < 1$ for all $x \neq y \in E$.

Lemma 2.3 (Simple norm estimate)**.**
For $x, y \in E$ with $x \neq y$ we have

$$
\left| \frac{|x|^2 - |y|^2}{2\, |A(x - y)|} \right| < 1 \,.
$$

Proof. By definition we have the inequalities $|A^{-1}x| < 1$ and $|A^{-1}y| < 1$ for any two points x and y in the ellipsoid E. From the Cauchy-Schwarz inequality and the triangle inequality we therefore obtain

$$\left| \frac{|x|^2 - |y|^2}{2\,|A\,(x-y)|} \right| = \frac{\left|\left\langle A^{-1}\,(x-y)\,,A\,(x-y)\right\rangle\right|}{2\,|A\,(x-y)|} \leq \frac{|A^{-1}x| + |A^{-1}y|}{2} < 1\,,$$

as we intended to show. □

Recall that G_n denotes the fundamental solution Laplacian and therefore satisfies $\Delta_x G_n\,(x,y) = \delta_n\,(x-y)$. A simple change of variables implies the following result, that we will also require in the following sections.

Lemma 2.4 (Fundamental solution of Δ_{Ax})**.**
For every $f \in C_0^\infty(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$, we have the identity

$$f\,(x) = \det(A)\,\Delta_{Ax} \int_{\mathbb{R}^n} f\,(y)\,G_n(Ax, Ay)\mathrm{d}y\,.$$

Proof. Using that G_n is the fundamental solution of Laplacian and making the coordinate change $u = Ay$ shows

$$\begin{aligned}\Delta_{Ax} \int_{\mathbb{R}^n} f\,(y)\,G_n(Ax, Ay)\mathrm{d}y &= \int_{\mathbb{R}^n} f\,(y)\,\delta_n(Ax - Ay)\mathrm{d}y \\ &= \frac{1}{\det(A)} \int_{\mathbb{R}^n} f\left(A^{-1}u\right)\delta_n(Ax-u)\mathrm{d}u = \frac{f\,(x)}{\det(A)}\,.\end{aligned}$$

Hence we have verified the required identity. □

Finally, we will also make use of the following well known fact, that the spherical means satisfy the Darboux equation:

Lemma 2.5 (Darboux equation)**.**
The spherical means of any $f \in C^\infty\,(\mathbb{R}^n)$ satisfy the Darboux equation

$$\Delta_x \mathcal{M}f\,(x,r) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)\mathcal{M}f\,(x,r) \qquad \textit{for } (x,r) \in \mathbb{R}^n \times (0,\infty)\,. \tag{2.4}$$

The right hand side in (2.4) may also be written as $(r^{1-n}\partial_r r^{n-1}\partial_r \mathcal{M}f)\,(x,r)$.

Proof. See, for example, [9]. □

3 Inversion in even dimension

Throughout this section, let $n \geq 2$ be a given even number. We will consider the function $\Phi_{\text{even}} \in L^1_{\text{loc}}\,(\mathbb{R})$ defined by

$$\Phi_{\text{even}}\,(s) := \log|s| \qquad \text{for } s \neq 0\,. \tag{3.1}$$

Further we denote by $\Phi_{\mathrm{even}}^{(n-2)}$ its $n-2$ fold distributional derivative. According to Lemma 2.2 we have, for the corresponding reconstruction integral defined by (2.1),

$$\begin{aligned}(\mathcal{B}_{\Phi_{\mathrm{even}}}\mathcal{M}f)(x) = \frac{\omega_{n-2}}{\omega_{n-1}} \int_{\mathbb{R}^n} f(y) \int_{-1}^{1} \left(1-s^2\right)^{(n-3)/2} \\ \times \Phi_{\mathrm{even}}^{(n-2)} \left(2\left|Ax-Ay\right| \left(\frac{|x|^2-|y|^2}{2\left|A(x-y)\right|} - s\right)\right) \mathrm{d}s\mathrm{d}r\,.\end{aligned}$$

The reconstruction formula in even dimension essentially follows from this identity after identifying the inner integral as a constant multiple of the fundamental solution of the Laplacian in Ax.

Lemma 3.1 (A Hilbert transform identity)**.**
Denote $\left(1-s^2\right)_+^{(n-3)/2} := \max\left\{0, 1-s^2\right\}^{(n-3)/2}$ *for* $s \in \mathbb{R}$ *and recall that* $\mathcal{H}_s[\,\cdot\,]$ *denotes the Hilbert transform. Then, for even* $n \geq 4$*, we have*

$$\partial_{s_\star}^{n-3}\mathcal{H}_s\left[\left(1-s^2\right)_+^{(n-3)/2}\right](s_\star) = (-1)^{n/2}(n-3)! \quad \text{for } |s_\star| < 1\,. \tag{3.2}$$

Proof. We first note the identity $\mathcal{H}_s[sg](s_\star) = s_\star\mathcal{H}_s[g](s_\star) - \frac{1}{\pi}\int_{\mathbb{R}} g(s)\,\mathrm{d}s$ for $g\colon \mathbb{R} \to \mathbb{R}$; see [34, Table 7.3]. Applying this identity repeatedly and using the fact that the function $\left(1-s^2\right)^{(n-3)/2}$ is equal to $\sqrt{1-s^2}$ times is a polynomial of degree $n-4$ yields

$$\mathcal{H}_s\left[\left(1-s^2\right)_+^{(n-3)/2}\right](s_\star) = Q_{n-4}(s_\star)\,\mathcal{H}_s\left[(1-s^2)_+^{1/2}\right](s_\star) + Q_{n-5}(s_\star)\;,$$

for a certain polynomials Q_{n-4} and Q_{n-5} of degree $n-4$ and $n-5$, respectively. (In the case $n=4$ we take $Q_{n-5}=0$.) The Hilbert transform of $(1-s^2)_+^{1/2}$ is known and equal to $s_\star$ on $\{|s_\star|<1\}$; see [6, Table 13.11]. This implies the identity

$$\mathcal{H}_s\left[\left(1-s^2\right)_+^{(n-3)/2}\right](s_\star) = c s_\star^{n-3} + P_{n-4}(s_\star) \quad \text{for } |s_\star| < 1\,,$$

where c is the leading coefficient of Q_{n-4} and P_{n-4} a polynomial of degree $\leq n-4$. The leading coefficient of Q_{n-4} equals the leading coefficient of $(1-s^2)^{(n-4)/2}$ and is given by $c = (-1)^{(n-4)/2} = (-1)^{n/2}$. Consequently, $\partial_{s_\star}^{n-3}\mathcal{H}_s\left[(1-s^2)_+^{(n-3)/2}\right]$ is constant for $|s_\star|<1$ and its value is $(-1)^{n/2}(n-3)!$. This shows (3.2). ☐

The following Lemma is the key to the reconstruction formula (1.3).

Lemma 3.2 (Kernel in even dimension)**.**
For every $x, y \in E$ *with* $x \neq y$ *and every* $s \in \mathbb{R}$ *with* $|s_\star| < 1$*, we have*

$$\begin{aligned}\frac{\omega_{n-2}}{\omega_{n-1}} \int_{-1}^{1} \left(1-s^2\right)^{(n-3)/2} \Phi_{\mathrm{even}}^{(n-2)}\left(2\left|Ax-Ay\right|(s_\star - s)\right)\mathrm{d}s \\ = (-1)^{(n-2)/2}\frac{\pi\omega_{n-2}(n-2)!}{2^{n-2}} G_n(Ax, Ay)\;.\end{aligned}$$

Proof. We first consider the case $n = 2$, where the integral to be computed is given by

$$\frac{1}{\pi}\int_{-1}^{1}\left(1-s^2\right)^{-1/2}\log\left|2\left|Ax-Ay\right|(s_\star - s)\right|\mathrm{d}s \\ = \log\left(2|Ax-Ay|\right) + \frac{1}{\pi}\int_{-1}^{1}\frac{\log|s_\star - s|}{\sqrt{1-s^2}}\mathrm{d}s\,.$$

Substituting $s = \cos\alpha$, the latter integral equals $\frac{1}{\pi}\int_0^\pi \log|s_\star - \cos(\alpha)|\,\mathrm{d}\alpha$. This integral has been computed in [13] and the result is $-\log 2$. Hence the above sum is $\log(2|Ax-Ay|) - \log 2 = \log|Ax-Ay|$. Since $G_2(x,y) = 1/(2\pi)\log|x-y|$ this shows the claim for the case $n = 2$.

Now suppose $n \geq 4$. After recalling that $\Phi_{\text{even}}(s) = \log|s|$, that the distributional derivative of Φ_{even} is $\mathrm{P.V.}(1/s)$, and that the Hilbert transform is defined as the convolution with $\mathrm{P.V.}(1/\pi s)$ we obtain

$$\begin{aligned}
&\frac{\omega_{n-2}}{\omega_{n-1}}\int_{-1}^{1}\left(1-s^2\right)^{(n-3)/2}\Phi_{\text{even}}^{(n-2)}\left(2\left|Ax-Ay\right|(s_\star - s)\right)\mathrm{d}s \\
&\qquad = \frac{\omega_{n-2}}{\omega_{n-1}2^{n-2}\left|Ax-Ay\right|^{n-2}}\partial_{s_\star}^{n-3}\int_{-1}^{1}\left(1-s^2\right)^{(n-3)/2}\frac{1}{s_\star - s}\mathrm{d}s \\
&\qquad = \frac{\omega_{n-2}\pi}{\omega_{n-1}2^{n-2}\left|Ax-Ay\right|^{n-2}}\partial_{s_\star}^{n-3}\mathcal{H}_s\left[\left(1-s^2\right)_+^{(n-3)/2}\right](s_\star) \\
&\qquad = (-1)^{n/2}\frac{\omega_{n-2}\pi(n-3)!}{\omega_{n-1}2^{n-2}\left|Ax-Ay\right|^{n-2}} \\
&\qquad = (-1)^{(n-2)/2}\,\frac{\omega_{n-2}\pi(n-2)!}{2^{n-2}}\,G_n(Ax,Ay)\;.
\end{aligned}$$

For the second last equality we used Equation (3.2) and the last identity follows from the identity $G_n(Ax,Ay) = -\left|Ax-Ay\right|^{2-n}/(\omega_{n-1}(n-2))$. The last displayed equation shows the desired identity for the case $n \geq 4$. □

3.1 Proof of Theorem 1.1

Lemmas 2.2, 2.3 and 3.2 show

$$\begin{aligned}
\left(\mathcal{B}_{\Phi_{\text{even}}}\mathcal{M}f\right)(x) &= \int_{S^{n-1}}\int_0^\infty r\mathcal{D}_r^{n-2}r^{n-2}\left(\mathcal{M}f\right)(A\sigma,r)\,\Phi_{\text{even}}\left(|x-A\sigma|^2 - r^2\right)\mathrm{d}r\mathrm{d}\sigma \\
&= \frac{\omega_{n-2}}{\omega_{n-1}}\int_{\mathbb{R}^n}f(y)\int_{-1}^{1}\left(1-s^2\right)^{(n-3)/2} \\
&\qquad\qquad \times\Phi_{\text{even}}^{(n-2)}\left(2\left|Ax-Ay\right|\left(\frac{|x|^2-|y|^2}{2\left|A(x-y)\right|} - s\right)\right)\mathrm{d}s \\
&= (-1)^{(n-2)/2}\frac{\pi\omega_{n-2}(n-2)!}{2^{n-2}}\int_{\mathbb{R}^n}f(y)\,G_n(Ax-Ay)\,\mathrm{d}y\,.
\end{aligned}$$

According to Lemma 2.4, application of the Laplacian in the variable Ax to the last integral gives $f(x)/\det(Ax)$. Consequently,

$$\begin{aligned}(-1)^{(n-2)/2}\frac{\pi\omega_{n-2}(n-2)!}{2^{n-2}\det(A)}f(x) &= \Delta_{Ax}\left(\mathcal{B}_{\Phi_{\rm even}}\mathcal{M}f\right)(x)\\ &= \Delta_{Ax}\int_{S^{n-1}}\int_0^\infty r\mathcal{D}_r^{n-2}r^{n-2}\left(\mathcal{M}f\right)(A\sigma,r)\log\left||x-A\sigma|^2-r^2\right|\mathrm{d}r\mathrm{d}\sigma\,,\end{aligned}$$

which is the desired reconstruction formula for the even dimensional case, stated in Theorem 1.1.

4 Inversion in odd dimension

Now let $n\geq 3$ be an odd number. The proof of the reconstruction formula of Theorem 1.2 will be similar to proof of the corresponding formula in the even dimensional case. However, the derivation of the reconstruction formula in odd dimensions is based on the use of the Heaviside function

$$\Phi_{\rm odd}\colon\mathbb{R}\to\mathbb{R}\colon s\mapsto\chi\left\{s>0\right\}$$

in place of the logarithmic function $\log|s|$ used in even dimensions.

The following key Lemma is the counterpart of Lemma 3.2 from the even dimensional case. It is however much easier to establish.

Lemma 4.1 (Kernel in odd dimension).
For every $x,y\in E$ with $x\neq y$ and every $s_\star\in\mathbb{R}$ with $|s_\star|<1$, we have

$$\begin{aligned}&\frac{\omega_{n-2}}{\omega_{n-1}}\int_{-1}^1\left(1-s^2\right)^{(n-3)/2}\Phi_{\rm odd}^{(n-2)}\left(2\left|Ax-Ay\right|(s_\star-s)\right)\mathrm{d}s\\ &\qquad\qquad= (-1)^{(n-1)/2}\frac{\omega_{n-2}(n-2)!}{2^{n-2}}G_n\left(Ax,Ay\right)\,.\end{aligned}$$

Proof. The first distributional derivative of $\Phi_{\rm odd}$ is given by the delta distribution, $\Phi_{\rm odd}'(s)=\delta(s)$. Consequently,

$$\begin{aligned}&\frac{\omega_{n-2}}{\omega_{n-1}}\int_{-1}^1\left(1-s^2\right)^{(n-3)/2}\Phi_{\rm odd}^{(n-2)}\left(2\left|Ax-Ay\right|(s_\star-s)\right)\mathrm{d}s\\ &\qquad= \frac{\omega_{n-2}}{\omega_{n-1}2^{n-2}\left|Ax-Ay\right|^{n-2}}\partial_{s_\star}^{n-3}\int_{-1}^1\left(1-s^2\right)^{(n-3)/2}\delta(s_\star-s)\mathrm{d}s\\ &\qquad= \frac{\omega_{n-2}}{\omega_{n-1}2^{n-2}\left|Ax-Ay\right|^{n-2}}\partial_{s_\star}^{n-3}\left(1-s_\star^2\right)^{(n-3)/2}\\ &\qquad= (-1)^{(n-3)/2}\frac{\omega_{n-2}(n-3)!}{\omega_{n-1}2^{n-2}\left|Ax-Ay\right|^{n-2}}\\ &\qquad= (-1)^{(n-1)/2}\frac{\omega_{n-2}(n-2)!}{2^{n-2}}G_n\left(Ax,Ay\right)\,.\end{aligned}$$

For the second last identity we made use of the fact that since n is odd, $(1-s_\star^2)^{(n-3)/2}$ is a polynomial of degree $n-3$ for $|s_\star| < 1$ and for the last identity we used the representation $G_n(x,y) = -|x-y|^{2-n}/(\omega_{n-1}(n-2))$ of the fundamental solution of the Laplacian in n spatial dimensions. $\square$

4.1 Proof of Theorem 1.2

Lemmas 2.2, 2.3 and 4.1 show

$$
\begin{aligned}
(\mathcal{B}_{\Phi_{\mathrm{odd}}}\mathcal{M}f)(x) &= \frac{\omega_{n-2}}{\omega_{n-1}} \int_{\mathbb{R}^n} f(y) \int_{-1}^{1} \left(1-s^2\right)^{(n-3)/2} \\
&\qquad \times \Phi_{\mathrm{odd}}^{(n-2)} \left(2\,|Ax - Ay| \left(\frac{|x|^2 - |y|^2}{2\,|A(x-y)|} - s \right) \right) \mathrm{d}s \\
&= (-1)^{(n-1)/2} \frac{\omega_{n-2}(n-2)!}{2^{n-2}} \int_{\mathbb{R}^n} f(y)\, G_n(Ax - Ay)\, \mathrm{d}y\,.
\end{aligned}
$$

Application of Lemma 2.4 and recalling that $\mathcal{D}_r^* r = -r\mathcal{D}_r$ yield

$$
\begin{aligned}
&(-1)^{(n-1)/2} \frac{\omega_{n-2}(n-2)!}{\det(A)\, 2^{n-2}} f(x) \\
&\quad = \Delta_{Ax} \left(\mathcal{B}_{\Phi_{\mathrm{odd}}}\mathcal{M}f\right)(x) \\
&\quad = \Delta_{Ax} \int_{S^{n-1}} \int_0^\infty r\mathcal{D}_r^{n-2} r^{n-2} (\mathcal{M}f)(A\sigma, r)\, \chi\left\{|x - A\sigma|^2 - r^2\right\} \mathrm{d}r\mathrm{d}\sigma \\
&\quad = \Delta_{Ax} \int_{S^{n-1}} \int_0^\infty r\mathcal{D}_r^{n-3} r^{n-2} (\mathcal{M}f)(A\sigma, r)\, \delta\left(|x - A\sigma|^2 - r^2\right) \mathrm{d}r\mathrm{d}\sigma \\
&\quad = \frac{1}{2} \Delta_{Ax} \int_{S^{n-1}} \int_0^\infty \mathcal{D}_r^{n-3} r^{n-2} (\mathcal{M}f)(A\sigma, r)\, \delta\left(|x - A\sigma| - r\right) \mathrm{d}r\mathrm{d}\sigma \\
&\quad = \frac{1}{2} \Delta_{Ax} \int_{S^{n-1}} \left(\mathcal{D}_r^{n-3} r^{n-2} \mathcal{M}f\right)(A\sigma, |x - A\sigma|)\, \mathrm{d}\sigma\,.
\end{aligned}
$$

This shows (1.4) and concludes the proof of Theorem 1.2.

5 Discussion

The problem of reconstructing a function from spherical means is important for many imaging and remote sensing applications (see, for example, [5, 7, 15, 22, 30, 31, 39]). Especially in the context of the novel hybrid imaging methods photoacoustic and thermoacoustic tomography many solution methods have been developed. Known reconstruction techniques can be classified in iterative reconstruction methods (see [10, 11, 33, 40, 42]), model based time reversal (see [8, 14, 20, 35]), Fourier domain algorithms (see [1, 19, 21, 24, 26, 31, 41]), and algorithms based on explicit reconstruction formulas of the back-projection type (see [3, 12, 14, 13, 16, 23, 25, 29, 32, 38]). The back-projection approach implements explicit solutions of the reconstruction problem. It is therefore much faster than iterative solution techniques, where the spherical mean transform and some adjoint transform have to be applied repeatedly.

Explicit back-projection type formulas for recovering a function from spherical mean values on a surface S are only known for certain type of surfaces. Such formulas have been derived for a planar surface in [3, 12] and much later for spherical surfaces in [14, 13, 23, 38]. In two and three spatial dimensions a formula for certain polygons and polyhedra have been derived in [25] and for ellipses and ellipsoids in [4, 17, 29]. Back-projection type formulas for ellipsoids in arbitrary dimension have been found in [18, 32]. In this paper we derived a new formula for recovering a function from spherical means with centers on an ellipsoid in arbitrary dimension (see Theorem 1.1) which is different from the ones in [18, 32]. Our reconstruction formula generalizes one of the formulas of [13, 14] from the spherical to the elliptical case and can be numerically implemented in an efficient way following, for example, the implementations presented in [2, 13] for spherical center sets.

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