Nr.03 / 2013

Leopold-Franzens-Universität Innsbruck

Preprint-Series: Department of Mathematics - Applied Mathematics

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APPLIEDMATHEMATICS

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UNIVERSAL INVERSION FORMULAS FOR RECOVERING A FUNCTION FROM SPHERICAL MEANS

MARKUS HALTMEIER*

Abstract. The problem of reconstruction a function from spherical means is at the heart of several modern imaging modalities and other applications. In this paper we derive universal back-projection type reconstruction formulas for recovering a function in arbitrary dimension from averages over spheres centered on the boundary an arbitrarily shaped bounded convex domain with smooth boundary. Provided that the unknown function is supported inside that domain, the derived formulas recover the unknown function up to an explicitly computed integral operator. For elliptical domains the integral operator is shown to vanish and hence we establish exact inversion formulas for recovering a function from spherical means centered on the boundary of elliptical domains in arbitrary dimension.

Key words. Spherical means, reconstruction formula, inversion formula, wave equation, universal back-projection, Radon transform, photoacoustic tomography, thermoacoustic tomography.

AMS subject classifications. 45Q05, 65J22, 65M32, 92C55, 35L05.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain in $\mathbb{R}^n$ with smooth boundary. In this paper we study the problem of recovering a function $f\colon \mathbb{R}^n \to \mathbb{R}$ that is supported in Ω from the averages (spherical means)

$$(\mathcal{M}f)(x,r) := \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + r\sigma)\, \mathrm{d}S(\sigma) \tag{1.1}$$

over spherical surfaces with centers $x \in \partial\Omega$ and radii $r > 0$. Here $S^{n-1} \subset \mathbb{R}^n$ is the $n-1$ dimensional unit sphere, ω_{n-1} its total surface area and $\mathrm{d}S$ denotes the standard surface measure. Further recall that a domain is an open, connected, nonempty set. We develop closed form reconstruction formulas of the backprojection type for recovering the function f from its spherical means $\mathcal{M}f(x,r)$ defined by (1.1). The derived formulas can be applied to arbitrarily shaped domains in arbitrary dimensions and recover the unknown function modulo an explicitly computed integral operator $\mathcal{K}_\Omega$. For elliptical domains, the operator $\mathcal{K}_\Omega$ is shown to vanish. We therefore establish exact reconstruction formulas of the backprojection type in these cases. Our results generalize the ones recently obtained in in [15] for $n = 3$ and [11] for $n = 2$ to the case of arbitrary spatial dimension.

The problem of recovering a function from spherical means is at the heart of many modern imaging applications, where the centers of the spheres of integration correspond to admissible locations of detectors recording some physical quantity encoded in f; see Figure 1.1. For example, recovering a function from spherical means is essential for the hybrid imaging techniques photoacoustic tomography (PAT) and thermoacoustic tomography (TAT) where the function f models the initial pressure of the acoustic field induced by a short electromagnetic pulse. In these applications the inversion from spherical means arises in three spatial dimensions (see [9, 12, 25]) as well as in two spatial dimensions in variants of PAT/TAT using integrating detectors (see [5, 20, 26]) instead of the more common point like detectors. In fact these applications initiated the authors interest in the problem of recovering a function from spherical means. The inversion from spherical means is, however, is also essential for other technologies such as SONAR (see [2, 22]), SAR imaging (see [1, 23]), ultrasound tomography (see [17, 18]), or seismic imaging (see [3, 6]).

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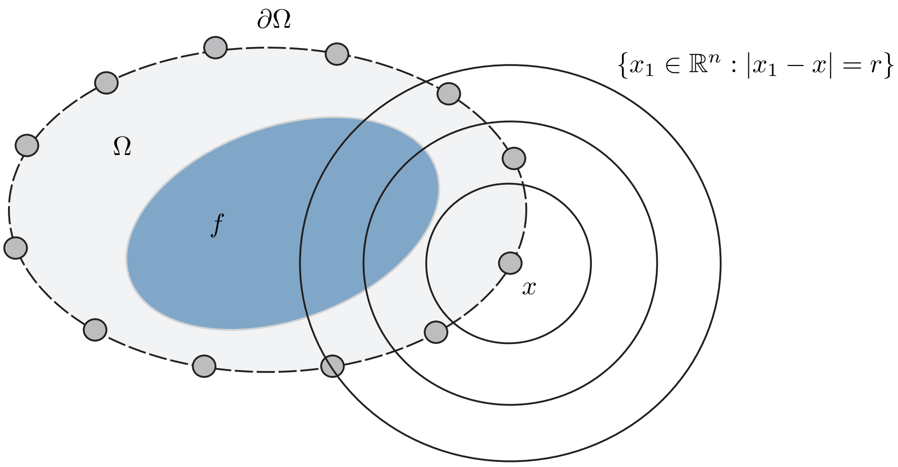

FIG. 1.1. Recovering a function from spherical means. *Suppose that the function f (representing some physical quantity of interest) is supported inside the domain Ω. Detectors are placed at various locations x on the boundary $\partial\Omega$ of the domain and record averages of f over spherical surfaces $\{x_1 \in \mathbb{R}^n : |x_1 - x| = r\}$ with radii $r > 0$. In this paper we derive explicit formulas for recovering the function f from these spherical averages (see Theorems 1.2, 1.3 and 1.4).*

1.1. Main results. Before presenting our main results, we introduce some notation. For any integrable function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, we define the Radon transform

$$(\mathcal{R}\varphi)(\omega, s) := \int_{\omega^\perp} \varphi(s\omega + y)\, \mathrm{d}S(y) \qquad \text{for} \quad (\omega, s) \in S^{n-1} \times \mathbb{R}\,,$$

where $\omega^\perp := \{y \in \mathbb{R}^n : \omega \cdot y = 0\}$ denotes the hyperplane consisting of all vectors orthogonal to $\omega \in S^{n-1}$. The derivative of a function $\psi\colon S^{n-1} \times \mathbb{R} \to \mathbb{R}$ in the second argument will be denoted by $(\partial_s \psi)(\omega, s)$, and $(\mathcal{H}_s \psi)(\omega, s)$ will be used to denote the Hilbert transform in the second argument (defined as the convolution with the principal value distribution $1/(\pi s)$). Further, for two distinct points $x_0, x_1 \in \mathbb{R}^n$, we set

$$\omega_\star(x_0, x_1) := \frac{x_1 - x_0}{|x_1 - x_0|}\,, \qquad s_\star(x_0, x_1) := \frac{1}{2}\,\frac{|x_1|^2 - |x_0|^2}{|x_1 - x_0|}\,. \tag{1.2}$$

As illustrated in Figure 1.2, the set $H_\star(x_0, x_1) = \{x \in \mathbb{R}^n : \omega_\star(x_0, x_1) \cdot x = s_\star(x_0, x_1)\}$ is the hyperplane of all points having the same distance between x_0 and x_1. The unit vector $\omega_\star(x_0, x_1)$ is orthogonal to the plane $H_\star(x_0, x_1)$ and $s_\star(x_0, x_1)$ is the oriented distance of that plane from the origin.

Inversion on general domains. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and denote by $\chi_\Omega\colon \mathbb{R}^n \to \mathbb{R}$ the characteristic function of Ω (taking the value one inside the domain Ω and the value zero outside). Further, for any C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ that is supported inside Ω and every $x_0 \in \Omega$, we define

$$(\mathcal{K}_\Omega f)(x_0) := \int_\Omega k_\Omega(x_0, x_1)\, f(x_1)\, \mathrm{d}x_1\,, \tag{1.3}$$

with

$$k_\Omega(x_0, x_1) := \begin{cases} \frac{(-1)^{(n-2)/2}}{2^{n+1}\pi^{n-1}} \, \frac{(\partial_s^n \mathcal{H}_s \mathcal{R}\chi_\Omega)(\omega_\star(x_0,x_1), s_\star(x_0,x_1))}{|x_1 - x_0|^{n-1}} & \text{if } n \text{ is even} \\[2ex] \frac{(-1)^{(n-1)/2}}{2^{n+1}\pi^{n-1}} \, \frac{(\partial_s^n \mathcal{R}\chi_\Omega)(\omega_\star(x_0,x_1), s_\star(x_0,x_1))}{|x_1 - x_0|^{n-1}} & \text{if } n \text{ is odd} \end{cases}\,. \tag{1.4}$$

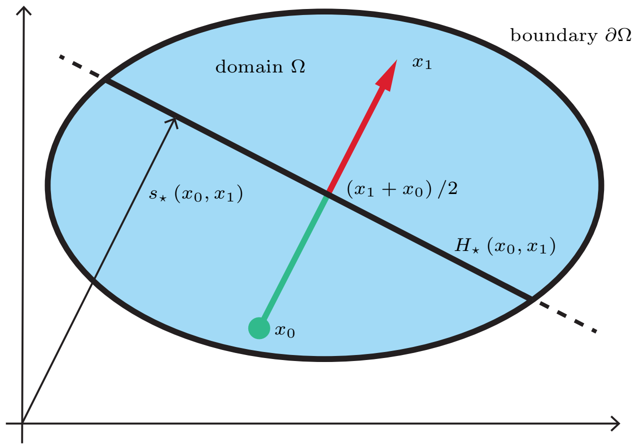

FIG. 1.2. Mid-plane between two distinct points x_0 and x_1 in $\mathbb{R}^n$. *The hyperplane* $H_\star(x_0, x_1) = \{x \in \mathbb{R}^n : \omega_\star(x_0, x_1) \cdot x = s_\star(x_0, x_1)\}$ *is the mid-plane between* x_1 *and* x_0*, that is, consists of all points having the same distance between these two points. The unit vector* $\omega_\star(x_0, x_1) = (x_1 - x_0)/|x_1 - x_0|$ *is orthogonal to the plane* $H_\star(x_0, x_1)$ *and* $s_\star(x_0, x_1) = \left(|x_1|^2 - |x_0|^2\right)/(2|x_1 - x_0|)$ *is the oriented distance of that plane from the origin.*

Here and in similar situations, ∂_s^n denotes the n-fold composition of the differentiation operator ∂_s. Note that $\omega_\star(x_0, x_1)$, $s_\star(x_0, x_1)$ and $k_\Omega(x_0, x_1)$ are only defined when $x_0 \neq x_1$.

REMARK 1.1. *Since* Ω *is assumed to be a convex domain with* C^∞ *boundary, the Radon transform of* χ_Ω *is a smooth function except for pairs* $(\omega, s) \in S^{n-1} \times \mathbb{R}$ *where the corresponding plane* $\{x \in \mathbb{R}^n : \omega \cdot x = s\}$ *is tangential to the boundary* $\partial\Omega$*. For two distinct points* x_0, x_1 *inside the domain* Ω*, the mid-plane* $H_\star(x_0, x_1) = \{x \in \mathbb{R}^n : \omega_\star(x_0, x_1) \cdot x = s_\star(x_0, x_1)\}$ *between these points is never tangential to the boundary of the domain (see Figure 1.2) and further the operators* ∂_s^n *and* $\mathcal{H}_s$ *preserve the locations of singularities. Consequently, for any compact subset* $K \subset \Omega$*, the functions* $(\partial_s^n \mathcal{H}_s \mathcal{R}\chi_\Omega)(\omega_\star(x_0, x_1), s_\star(x_0, x_1))$ *and* $(\partial_s^n \mathcal{R}\chi_\Omega)(\omega_\star(x_0, x_1), s_\star(x_0, x_1))$ *are* C^∞ *and bounded on* $\{(x_0, x_1) \in K \times K : x_0 \neq x_1\}$*. This shows, that* k_Ω *is a weakly singular kernel on* $K \times K$ *and that the integral operator* $\mathcal{K}_\Omega : L^2(K) \to L^2(K)$ *is well defined and compact.*

The inversion formulas we establish in this paper are exact modulo the integral operator $\mathcal{K}_\Omega$. They look somewhat different in even and in odd dimensions and are stated in separate theorems below.

In even dimensions, our main result is as follows.

THEOREM 1.2 (Inversion in even dimension). *Let* $n \geq 2$ *be an even natural number, let* $\Omega \subset \mathbb{R}^n$ *be a bounded convex domain with smooth boundary* $\partial\Omega$*, and let* $f : \mathbb{R}^n \to \mathbb{R}$ *be* C^∞ *and supported inside* Ω*.*

Then, for every $x_0 \in \Omega$,

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-2)/2}\omega_{n-1}}{2\pi^n} \times \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_0^\infty \frac{\left(r\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x,r)}{r^2 - |x_0 - x|^2} \,\mathrm{d}r \mathrm{d}S(x) \,, \tag{1.5}$$

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-2)/2}\omega_{n-1}}{2\pi^n} \times \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \int_0^\infty \frac{\left(\partial_r\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x,r)}{r^2 - |x_0 - x|^2} \,\mathrm{d}r \mathrm{d}S(x) \,. \tag{1.6}$$

In both formulas, the inner integration is taken in the principal value sense, $\mathrm{d}S$ *is the usual surface measure,* ν_x *is the outward pointing unit normal to* Ω*, and* $\mathcal{K}_\Omega$ *is the integral operator defined by (1.3), (1.4). Moreover,* $\mathcal{D}_r := (2r)^{-1}\partial_r$ *denotes differentiation with respect to* r^2 *and* $\nabla_{x_0}\cdot$ *the divergence with respect to* x_0.

Proof. See Section 3. ∎

In odd dimension we have the following corresponding result.

THEOREM 1.3 (Inversion in odd dimension). *Let* $n \geq 3$ *be an odd natural number, let* $\Omega \subset \mathbb{R}^n$ *be a bounded convex domain with smooth boundary* $\partial\Omega$*, and let* $f \colon \mathbb{R}^n \to \mathbb{R}$ *be* C^∞ *and supported inside* Ω.

Then, for every $x_0 \in \Omega$,

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-3)/2}\omega_{n-1}}{4\pi^{n-1}} \times \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x, |x_0 - x|) \,\mathrm{d}S(x) \,, \tag{1.7}$$

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-3)/2}\omega_{n-1}}{4\pi^{n-1}} \times \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \left(\partial_r\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x, |x_0 - x|) \,\mathrm{d}S(x) \,. \tag{1.8}$$

Here $\mathcal{K}_\Omega$, ν_x, ∇_{x_0}, $\mathrm{d}S$, *and* $\mathcal{D}_r$ *are as in Theorem 1.2.*

Proof. See Section 4. ∎

Both, Theorem 1.2 and Theorem 1.3 will follow from corresponding statements for the inversion of the wave equation in even and odd dimensions, which we shall establish in the following sections (see Theorems 3.1 and 4.1).

Exact reconstruction for elliptical domains. In the case that Ω is an elliptical domain, we show that the integral operator $\mathcal{K}_\Omega$ vanishes exactly and therefore Theorems 1.2 and 1.3 provide exact reconstruction formulas for ellipsoids. After translation and rotation, we may assume that the elliptical domain takes the standard form

$$\Omega := \left\{ x \in \mathbb{R}^n : \left|A^{-1}x\right|^2 < 1 \right\} \,,$$

where $A \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive (possibly distinct) diagonal entries. Obviously, a ball is a special case of an elliptical domain where all diagonal entries of A coincide and are equal to the radius of the ball.

For elliptical domains we have the following exact inversion formulas.

THEOREM 1.4 (Exact inversion on elliptical domains). *Let* $\Omega \subset \mathbb{R}^n$ *be an elliptical domain. Then* $\mathcal{K}_\Omega f$ *vanishes identically on* Ω*. In particular, for any smooth function* $f \colon \mathbb{R}^n \to \mathbb{R}$ *that is supported inside* Ω *and every* $x_0 \in \Omega$*, the following hold:*

(a) If n is even, then

$$f(x_0) = \frac{(-1)^{(n-2)/2}\omega_{n-1}}{2\pi^n} \times \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_0^\infty \frac{\left(r\mathcal{D}_r^{n-2} r^{n-2}\mathcal{M}f\right)(x,r)}{r^2 - |x_0 - x|^2} \, \mathrm{d}r \mathrm{d}S(x) \,, \tag{1.9}$$

$$f(x_0) = \frac{(-1)^{(n-2)/2}\omega_{n-1}}{2\pi^n} \times \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \int_0^\infty \frac{\left(\partial_r \mathcal{D}_r^{n-2} r^{n-2}\mathcal{M}f\right)(x,r)}{r^2 - |x_0 - x|^2} \, \mathrm{d}r \mathrm{d}S(x) \,. \tag{1.10}$$

(b) If n is odd, then

$$f(x_0) = \frac{(-1)^{(n-3)/2}\omega_{n-1}}{4\pi^{n-1}} \times \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_r^{n-2} r^{n-2}\mathcal{M}f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \,, \tag{1.11}$$

$$f(x_0) = \frac{(-1)^{(n-3)/2}\omega_{n-1}}{4\pi^{n-1}} \times \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \left(\partial_r \mathcal{D}_r^{n-2} r^{n-2}\mathcal{M}f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \,. \tag{1.12}$$

Here ν_x, ∇_{x_0}, $\mathrm{d}S$, and $\mathcal{D}_r$ are as in Theorem 1.2.

Proof. See Section 5. □

By taking limits one can easily establish exact inversion formulas like (1.9)–(1.12) for certain unbounded domains, such as for elliptical cylinders. We omit formulating such generalizations. Further, it would be interesting so find further domains Ω, where the integral operator $\mathcal{K}_\Omega$ can be shown to vanish. Such an investigation, however, is beyond the scope of this paper.

1.2. Relations to previous work. Exact back-projection type inversion formulas for recovering a function from spherical means with centers on the boundary of a ball have been discovered quite recently in [7, 8, 13, 16, 24]. In [24] a formula has been found for $n = 3$, which has later been generalized to arbitrary dimensions in [13]. In odd dimension these formulas coincides with our Equation (1.11) (which, however, holds for the more general case of elliptical center sets). A different set of exact reconstruction formulas has been derived in [8] for odd dimensions and in [7] for even dimensions. In [9] relations between the different formulas have been investigated for dimensions $n = 2$ and $n = 3$. None of these papers considers the case of more general domains. In [14] reconstruction formulas of the back-projection type have been found for certain polygons and polyhedra in two and three spatial dimensions.

Formulas that recover a function from spherical means with centers on the boundary of an elliptical domain in arbitrary dimension have been obtained in [19, Equations (20), (21)]. The derived identities as well as the method of proof are different from ours. Our results are, however, closely related to ones of [11, 15]. Actually, the present article generalizes the result obtained for $n = 2$ in [11] and for $n = 3$ in [15] to the case of arbitrary spatial dimension.

1.3. Outline. The main aim of the following sections is the proof of Theorems 1.2, 1.3 and 1.4. To that end, we first derive an auxiliary identity for the wave equation in Section 2 (see Theorem 2.3). Subsequently, in Section 3 we shall prove Theorem 1.2 and in Section 4 we establish Theorem 1.3. In these sections, we also derive corresponding statement for recovering the initial data of the wave equation from the solution on the boundary of an arbitrarily shaped domain. These results, which are also of interest in their own, will be presented in Theorems 3.1 and 4.1 below. In Section 5 we consider the case of elliptical domains, where we show that the operator $\mathcal{K}_\Omega$ vanishes identically and therefore we establish the exact reconstruction formulas stated in Theorem 1.4. The paper concludes with a discussion in Section 6.

2. Auxiliary results for the wave equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function that is supported inside Ω. Consider the following initial value problem for the wave equation

$$\begin{cases} \left(\partial_t^2 - \Delta_x\right) p\,(x,t) = 0\,, & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ p\,(x,0) = f\,(x)\,, & \text{for } x \in \mathbb{R}^n \\ \partial_t p\,(x,0) = 0\,, & \text{for } x \in \mathbb{R}^n \end{cases}. \tag{2.1}$$

Here ∂_t denotes differentiation with respect to the temporal variable $t \in (0,\infty)$ and Δ_x is the Laplacian in the spatial variable $x \in \mathbb{R}^n$. To indicate the dependance of the initial data we will also write the solution of (2.1) as $p = \mathcal{W}f$.

According to the well known explicit formulas for the solution of (2.1) in terms of spherical means, recovering a function from spherical means is essentially equivalent to the problem of recovering the initial data in (2.1) from values of the solution on $\partial\Omega\times(0,\infty)$. In this section we derive a basic result for the wave inversion which in the following sections will be applied to derive the results for the inversion form spherical means presented in the introduction.

2.1. Outgoing fundamental solution. Throughout the following we denote by $G\,(x,t)$ the outgoing fundamental solution (or free space Green's function) of the wave equation, that vanishes on $\{t<0\}$ and satisfies the equation

$$\left(\partial_t^2 - \Delta_x\right) G\,(x,t) = \delta_n\,(x)\,\delta\,(t)\,, \qquad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}\,.$$

Here and in the following, δ_n and δ denote the n-dimensional and one-dimensional delta distribution, respectively.

Remark 2.1. *By definition, the outgoing fundamental solution is a distribution on $\mathbb{R}^n \times \mathbb{R}$. The arguments in $G\,(x,t)$ do not mean a point-evaluation at (x,t) but are only a formal notation indicating the variables, where this distributions acts on. Derivatives of the fundamental solution, like $\partial_t G\,(x,t)$, will always denote distributional derivatives. Further, notice that the mapping $t \mapsto G\,(\,\cdot\,,t)$ from $(0,\infty)$ to the space $\mathcal{D}'\,(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n$ is well defined and C^∞.*

With the outgoing fundamental solution of the wave equation, the solution of the initial value problem (2.1) can be written as

$$p\,(x,t) = \int_\Omega \partial_t G\,(x - x_1, t)\, f\,(x_1)\,\mathrm{d}x_1\,, \qquad \text{for all } (x,t) \in \mathbb{R}^n \times (0,\infty)\,. \tag{2.2}$$

Throughout this paper integrals like the one on the right hand side in (2.2) will always be understood in the weak sense (for any fixed t). Hence, identity (2.2) actually means, that

$$\int_{\mathbb{R}^n} p\,(x,t)\,\varphi\,(x)\,\mathrm{d}x = \int_\Omega \langle \partial_t G\,(\,\cdot\, - x_1, t)\,, \varphi\rangle\, f\,(x_1)\,\mathrm{d}x_1 \tag{2.3}$$

for any smooth test function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ with $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing between a distribution and a test function on $\mathbb{R}^n$.

After inserting the known explicit expressions for the outgoing fundamental solution G, Equation (2.2) can be rewritten in terms of spherical means of the function f. The explicit solution formulas differ in even and in odd dimensions and will be stated in Sections 3 and 4, where we study the even and the odd dimensional case separately and in more detail.

2.2. Kirchhoff integral representation. The following Kirchhoff integral representation relates the initial conditions of the free space wave equation (2.1) with boundary values on some domain. It is well known for three spatial dimension (and follows, for example, from [10, Equation (4.1.25)]) but we did not found a reference for the case of arbitrary dimension. Since the Kirchhoff integral representation serves as the basis of our further computations, we decided to include a simple derivation based on Greens second identity.

Lemma 2.2 (Kirchhoff integral representation). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, let $p = \mathcal{W}f$ denote the solution of (2.1) with initial data $f \in C_c^\infty(\Omega)$, and let G denote the outgoing fundamental solution of the wave equation. Then, for every $x_0 \in \Omega$, we have*

$$f(x_0) = \int_{\partial\Omega} \nu_x \cdot \int_{\mathbb{R}} G(x_0 - x, t)\, \nabla_x p(x,t)\, \mathrm{d}t\mathrm{d}S(x) \\ - \int_{\partial\Omega} \nu_x \cdot \int_{\mathbb{R}} \nabla_x G(x_0 - x, t)\, p(x,t)\, \mathrm{d}t\mathrm{d}S(x)\,. \tag{2.4}$$

Proof. Greens second identity applied with $G(x_0 - \,\cdot\,, t)$ and $p(\,\cdot\,, t)$ for fixed $(t, x_0) \in (0,\infty) \times \Omega$ yields

$$\begin{aligned}\int_{\partial\Omega} \nu_x \cdot \left(G(x_0 - x, t)\, \nabla_x p(x,t) - p(x,t)\, \nabla_x G(x_0 - x, t)\right) \mathrm{d}S(x) \\ = \int_\Omega \left(G(x_0 - x, t)\, \Delta_x p(x,t) - p(x,t)\, \Delta_x G(x_0 - x, t)\right) \mathrm{d}x \\ = \int_\Omega \left(G(x_0 - x, t)\, \partial_t^2 p(x,t) - p(x,t)\, \partial_t^2 G(x_0 - x, t)\right) \mathrm{d}x\,.\end{aligned}$$

For the second equality we used the assumption, that $G(x_0 - \,\cdot\,, t)$ and $p(\,\cdot\,, t)$ both satisfy the wave equation on $\{t > 0\}$. Integrating the above identity over some finite time interval $[T_1, T_2] \subset (0,\infty)$, interchanging the order of integration, and performing two integration by parts gives

$$\begin{aligned}\int_{T_1}^{T_2} \int_{\partial\Omega} \nu_x \cdot \left(G(x_0 - x, t)\, \nabla_x p(x,t) - p(x,t)\, \nabla_x G(x_0 - x, t)\right) \mathrm{d}S(x)\,\mathrm{d}t \\ = \int_\Omega G(x_0 - x, T_2)\, \partial_t p(x, T_2)\, \mathrm{d}x - \int_\Omega p(x, T_2)\, \partial_t G(x_0 - x, T_2)\, \mathrm{d}x \\ - \int_\Omega G(x_0 - x, T_1)\, \partial_t p(x, T_1)\, \mathrm{d}x + \int_\Omega p(x, T_1)\, \partial_t G(x_0 - x, T_1)\, \mathrm{d}x\,.\end{aligned}$$

Since any solution of the wave equation with compact support tends to zero as $t \to \infty$ (uniformly on every bounded set), the first two terms term on the right hand vanish as $T_2 \to \infty$. Next, we note that by Duhamel's principle, we have $\lim_{t\to 0} G(x_0 - \,\cdot\,, t) = 0$ and $\lim_{t\to 0} \partial_t G(x_0 - \,\cdot\,, t) = \delta(x_0 - \,\cdot\,)$. Hence the latter difference converges to $f(x_0)$ as $T_1 \to 0$, which yields the claimed representation (2.4). □

In the proof of Lemma 2.2 as well as in the following derivations we formally operate with distributions as they were classical functions. These computations can be made more rigorous by writing down all equalities in the weak sense (similar as done in [11] for the two dimensional case) and then using classical integral calculus. However, the calculus using distributions used in the present paper seems to be more intuitive and easier to follow and has also been used in [8, 15] for the derivation of inversion formulas in three spatial dimensions.

2.3. Universal backprojection. For any smooth function $v\colon \partial\Omega \times (0,\infty) \to \mathbb{R}$ we define

$$(\mathcal{B}_\Omega v)(x_0) := 2\,\nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_{\mathbb{R}} G(x - x_0, t)\, v(x,t)\, \mathrm{d}t \mathrm{d}S(x)\,, \qquad \text{for } x_0 \in \Omega\,. \tag{2.5}$$

Note that in the even dimensional case, the function $v(x,t)$ needs some decay as $t \to \infty$ in order that the integral $(\mathcal{B}_\Omega v)(x_0)$ is well defined. This is certainly the case if v is the restriction of the solution of the initial value problem (2.1) for some compactly supported initial data f, which happens in all instances where we apply the operator $\mathcal{B}_\Omega$. In three spatial dimensions (and in a slightly different form), the inversion integral (2.5) has been introduced to photoacoustic tomography in [24]. According to the notion of [24], we call $\mathcal{B}_\Omega$ the *universal backprojection operator.*

In [24] it has been shown that the identity $\mathcal{B}_\Omega \mathcal{W} f = f$ holds for the case that Ω is an open ball in three spatial dimensions and that f is supported inside Ω. This exact reconstruction property does not hold for general domains. However, for arbitrarily shaped domains in arbitrary dimensions we have the following result:

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary. Then, for any C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ with support in Ω and any $x_0 \in \Omega$, we have*

$$f(x_0) = (\mathcal{B}_\Omega \mathcal{W} f)(x_0) + \int_\Omega f(x_1)\, k_\Omega^{(2)}(x_0, x_1)\, \mathrm{d}x_1\,, \tag{2.6}$$

with the distributional kernel

$$k_\Omega^{(2)}(x_0, x_1) := (\nabla_{x_0} + \nabla_{x_1})^2 \int_\Omega \int_{\mathbb{R}} (\partial_t G(x_1 - x, t))\, G(x_0 - x, t)\, \mathrm{d}t\, \mathrm{d}x\,. \tag{2.7}$$

Here $(\nabla_{x_0} + \nabla_{x_1})^2$ is s shorthand notation for the operator $(\nabla_{x_0} + \nabla_{x_1}) \cdot (\nabla_{x_0} + \nabla_{x_1})$ and the equations (2.6) and (2.7) have to be read in the weak sense.

Proof. According to Kirchhoff's integral representation (see Lemma 2.2), the solution $p = \mathcal{W} f$ of the initial value problem (2.1) satisfies

$$\begin{aligned} f(x_0) = &\int_{\partial\Omega} \nu_x \cdot \int_{\mathbb{R}} G(x_0 - x, t)\, \nabla_x p(x,t)\, \mathrm{d}t \mathrm{d}S(x) \\ &- \int_{\partial\Omega} \nu_x \cdot \int_{\mathbb{R}} p(x,t)\, \nabla_x G(x_0 - x, t)\, \mathrm{d}t \mathrm{d}S(x)\,. \end{aligned}$$

Now inserting the identity $G \nabla_x p = \nabla_x (pG) - p \nabla_x G$ in the first term followed by an application of the divergence theorem, and using the relation $\nabla_x G(x_0 - x, t) = -\nabla_{x_0} G(x_0 - x, t)$ yield

$$\begin{aligned} f(x_0) = &\int_\Omega \int_{\mathbb{R}} \Delta_x (p(x,t)\, G(x_0 - x, t))\, \mathrm{d}t \mathrm{d}x \\ &+ 2\,\nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_{\mathbb{R}} p(x,t)\, G(x_0 - x, t)\, \mathrm{d}t \mathrm{d}S(x)\,. \end{aligned} \tag{2.8}$$

According to the definition of $\mathcal{B}_\Omega$, the second term equals $(\mathcal{B}_\Omega \mathcal{W} f)(x_0)$. After inserting the representation (2.2) for the solution $p = \mathcal{W} f$ of the initial value problem (2.1) and applying the relation $\nabla_x G(x_i - x, t) = -\nabla_{x_i} G(x_i - x, t)$ with $i = 0, 1$, the first term in (2.8) is seen to take the form $\int_\Omega k_\Omega^{(2)}(x_0, x_1) f(x_1)\, \mathrm{d}x_1$, with $k_\Omega^{(2)}$ defined by Equation (2.7). This concludes the proof of Theorem 2.3. ◻

3. Inversion in even dimension. Now let $n \geq 2$ denote an even natural number. In this section we derive explicit formulas for the wave inversion in even dimension and then apply these results for establishing Theorem 1.2. In even dimension, the outgoing fundamental solution of the wave equation takes the following explicit form

$$G(x,t) = \begin{cases} \frac{1}{2\pi^{n/2}} \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2-|x|^2>0\right\}}{\sqrt{t^2-|x|^2}} & \text{on } \{t>0\} \\ 0 & \text{on } \{t<0\} \end{cases}, \tag{3.1}$$

where $\mathcal{D}_t = (2t)^{-1}\partial_t$ denotes differentiation with respect to t^2, and $\chi\left\{t^2 - |x|^2 > 0\right\}$ is the characteristic function of the set all points $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ with $t^2 - |x|^2 > 0$. We emphasize again that in (3.1) and in similar situations all derivatives are understood as distributional derivatives.

We now have the following result for recovering the initial data of the initial value problem (2.1) from the restriction of its solution to $\partial\Omega \times (0,\infty)$.

Theorem 3.1 (Wave inversion in even dimension). *Let $n \geq 2$ be an even natural number, let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function that is supported inside Ω. Then, for every $x_0 \in \Omega$,*

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-2)/2}}{\pi^{n/2}} \times \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_{|x_0-x|}^\infty \frac{\left(t\mathcal{D}_t^{(n-2)/2} t^{-1}\mathcal{W}f\right)(x,t)\,\mathrm{d}t}{\sqrt{t^2-|x_0-x|^2}}\,\mathrm{d}S(x)\,, \tag{3.2}$$

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-2)/2}}{\pi^{n/2}} \times \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \int_{|x_0-x|}^\infty \frac{\left(\partial_t\mathcal{D}_t^{(n-2)/2} t^{-1}\mathcal{W}f\right)(x,t)\,\mathrm{d}t}{\sqrt{t^2-|x_0-x|^2}}\,\mathrm{d}S(x)\,. \tag{3.3}$$

Here $\mathcal{K}_\Omega$, ν_x, ∇_{x_0}, and $\mathrm{d}S$ are as in Theorem 1.2.

We proceed this section by first deriving Theorem 3.1 and then establishing the corresponding result for the inversion from spherical means in even dimensions (namely Theorem 1.2).

3.1. Proof of Theorem 3.1. According to Theorem 2.3 we have to show that the kernel $k_\Omega^{(2)}$ defined in (2.7) is equal to the kernel k_Ω defined in (1.4), and that $(\mathcal{B}_\Omega\mathcal{W}f)(x_0)$ can be written as the integral term in (3.2) as well as the one in (3.3).

Let us start by showing that $k_\Omega^{(2)} = k_\Omega$, that is,

$$(\nabla_{x_0} + \nabla_{x_1})^2 \int_\Omega \int_{\mathbb{R}} (\partial_t G(x_1 - x, t))\, G(x_0 - x, t)\,\mathrm{d}t\mathrm{d}x = \frac{(-1)^{(n-2)/2}}{2^{n+1}\pi^{n-1}|x_1 - x_0|^{n-1}} (\partial_s^n \mathcal{H}_s \mathcal{R}\chi_\Omega)(\omega_\star, s_\star)\,, \tag{3.4}$$

where $\omega_\star = \omega_\star(x_0,x_1) = \frac{x_1-x_0}{|x_1-x_0|}$ and $s_\star = s_\star(x_0,x_1) = \frac{|x_1|^2-|x_0|^2}{2|x_1-x_0|}$ are as in (1.2).

To show (3.4), for any two given points $x_0 \neq x_1 \in \Omega$, we write $R_0 := |x_0 - x|$ and $R_1 := |x_1 - x|$. Then, using the explicit expression (3.1) for the fundamental solution of the wave equation in even dimensions, the inner integral on the left hand side of (3.4) evaluates to

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(\partial_t G\left(x_1 - x, t\right)\right) G\left(x_0 - x, t\right) \mathrm{d}t \\
&\qquad = \frac{1}{4\pi^n} \int_0^\infty \left(\partial_t \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 - R_1^2 > 0\right\}}{\sqrt{t^2 - R_1^2}} \right) \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 - R_0^2 > 0\right\}}{\sqrt{t^2 - R_0^2}} \,\mathrm{d}t \\
&\qquad = \frac{1}{4\pi^n} \int_0^\infty \left(\partial_t \mathcal{D}_{R_1}^{(n-2)/2} \frac{\chi\left\{t^2 - R_1^2 > 0\right\}}{\sqrt{t^2 - R_1^2}} \right) \mathcal{D}_{R_0}^{(n-2)/2} \frac{\chi\left\{t^2 - R_0^2 > 0\right\}}{\sqrt{t^2 - R_0^2}} \,\mathrm{d}t \\
&\qquad = \frac{1}{4\pi^n} \mathcal{D}_{R_1}^{(n-2)/2} \mathcal{D}_{R_0}^{(n-2)/2} \int_0^\infty \left(\partial_t \frac{\chi\left\{t^2 - R_1^2 > 0\right\}}{\sqrt{t^2 - R_1^2}} \right) \frac{\chi\left\{t^2 - R_0^2 > 0\right\}}{\sqrt{t^2 - R_0^2}} \,\mathrm{d}t \\
&\qquad = -\frac{2}{4\pi^n} \mathcal{D}_{R_1}^{(n-2)/2} \mathcal{D}_{R_0}^{(n-2)/2} \lim_{T\to\infty} \mathcal{D}_{R_1} \int_{\max\{R_0, R_1\}}^T \frac{t \,\mathrm{d}t}{\sqrt{t^2 - R_1^2}\sqrt{t^2 - R_0^2}} \,.
\end{aligned}
$$

For $T \geq \max\{R_0, R_1\}$, the above integral on the right hand side computes to

$$
\begin{aligned}
&\int_{\max\{R_0, R_1\}}^T \frac{t \,\mathrm{d}t}{\sqrt{t^2 - R_1^2}\sqrt{t^2 - R_0^2}} \\
&\qquad = \ln\left(\sqrt{T^2 - R_0^2} + \sqrt{T^2 - R_1^2} \right) - \frac{1}{2} \ln\left(\left| R_0^2 - R_1^2 \right| \right) . \qquad (3.5)
\end{aligned}
$$

After applying the operator $\mathcal{D}_{R_1}$ and letting $T \to \infty$, the first term vanishes. Now let $\Phi(s) = 1/s$ denote the principal value distribution $\varphi \mapsto \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \varphi(s) s^{-1} \mathrm{d}s$. Recalling the definitions of R_0 and R_1 then implies

$$
\begin{aligned}
\int_{\mathbb{R}} \left(\partial_t G\left(x_1 - x, t\right)\right) G\left(x_0 - x, t\right) \mathrm{d}t &= -\frac{1}{4\pi^n} \mathcal{D}_{R_1}^{(n-2)/2} \mathcal{D}_{R_0}^{(n-2)/2} \frac{1}{R_0^2 - R_1^2} \\
&= \frac{(-1)^{n/2}}{4\pi^n} \Phi^{(n-2)} \left(\left|x_0 - x\right|^2 - \left|x_1 - x\right|^2 \right) . \qquad (3.6)
\end{aligned}
$$

Here and in the following $\Phi^{(\nu)}$ denotes the ν-th distributional derivative of Φ for some integer number $\nu \geq 0$.

For the following recall that $\omega_\star = \frac{x_1 - x_0}{|x_1 - x_0|}$ and $s_\star = \frac{|x_1|^2 - |x_0|^2}{2|x_1 - x_0|}$ and write any point $x \in \mathbb{R}^n$ in the form $x = s\omega_\star + y$ with $s \in \mathbb{R}$ and $y \perp \omega_\star$. We then can compute

$$
\begin{aligned}
&\left|x_0 - x\right|^2 - \left|x_1 - x\right|^2 = \left|x_0\right|^2 - \left|x_1\right|^2 + 2x \cdot \left(x_1 - x_0\right) \\
&\qquad = 2\left(x_1 - x_0\right) \cdot \left(x - \frac{x_1 - x_0}{2} \right) = 2\left|x_1 - x_0\right| \left(s - s_\star\right) . \qquad (3.7)
\end{aligned}
$$

Together with Equation (3.6) and the definition of the Radon transform $\mathcal{R}$, this further implies

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}} \left(\partial_t G\left(x_1 - x, t\right)\right) G\left(x_0 - x, t\right) \mathrm{d}t \mathrm{d}x \\
&\qquad = \frac{(-1)^{n/2}}{4\pi^n} \int_{\mathbb{R}} \int_{\omega_s tar^\perp} \chi_\Omega\left(s\omega_\star + y\right) \Phi^{(n-2)} \left(2\left|x_1 - x_0\right| \left(s - s_\star\right)\right) \mathrm{d}y \,\mathrm{d}s \\
&\qquad = \frac{(-1)^{n/2}}{4\pi^n} \int_{\mathbb{R}} \Phi^{(n-2)} \left(2\left|x_1 - x_0\right| \left(s - s_\star\right)\right) \left(\int_{\omega_\star^\perp} \chi_\Omega\left(s\omega_\star + y\right) \,\mathrm{d}y \right) \mathrm{d}s \\
&\qquad = \frac{(-1)^{n/2}}{4\pi^n} \int_{\mathbb{R}} \Phi^{(n-2)} \left(2\left|x_1 - x_0\right| \left(s - s_\star\right)\right) \left(\mathcal{R}\chi_\Omega\right) \left(\omega_\star, s\right) \mathrm{d}s \,.
\end{aligned}
$$

Now using the chain rule, integrating by parts $n-2$ times, recalling the definition of the principal value distribution $\Phi(s) = 1/s$, and noting that the Hilbert transform $\mathcal{H}_s$ is defined as the convolution with $\pi^{-1}\Phi$ imply

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}} \left(\partial_t G(x_1 - x, t)\right) G(x_0 - x, t)\, \mathrm{d}t\mathrm{d}x \\
&\qquad = \frac{(-1)^{n/2}}{4\pi^n 2^{n-2} |x_1 - x_0|^{n-2}} \int_{\mathbb{R}} \left(\partial_s^{n-2}\Phi\left(2|x_1 - x_0|(s - s_\star)\right)\right)(\mathcal{R}\chi_\Omega)(\omega_\star, s)\, \mathrm{d}s \\
&\qquad = \frac{(-1)^{n/2}}{\pi^n 2^n |x_1 - x_0|^{n-2}} \int_{\mathbb{R}} \frac{1}{2|x_1 - x_0|(s - s_\star)} \left(\partial_s^{n-2}\mathcal{R}\chi_\Omega\right)(\omega_\star, s)\, \mathrm{d}s \\
&\qquad = \frac{(-1)^{(n-2)/2}}{2^{n+1}\pi^n |x_1 - x_0|^{n-1}} \int_{\mathbb{R}} \frac{1}{s_\star - s} \left(\partial_s^{n-2}\mathcal{R}\chi_\Omega\right)(\omega_\star, s)\, \mathrm{d}s \\
&\qquad = \frac{(-1)^{(n-2)/2}}{2^{n+1}\pi^{n-1} |x_1 - x_0|^{n-1}} \left(\partial_s^{n-2}\mathcal{H}_s\mathcal{R}\chi_\Omega\right)(\omega_\star, s_\star)\ .
\end{aligned}$$

It remains to apply the operator $\left(\nabla_{x_0} + \nabla_{x_1}\right)^2$ to the last expression. To that end, notice that due to symmetry $\nabla_{x_0} + \nabla_{x_1}$ applied to any distribution only depending on $x_0 - x_1$ vanishes, and that $\left(\nabla_{x_0} + \nabla_{x_1}\right) s_\star = (x_1 - x_0) / |x_1 - x_0|$. This implies

$$\begin{aligned}
&\left(\nabla_{x_0} + \nabla_{x_1}\right)^2 \int_\Omega \int_{\mathbb{R}} \left(\partial_t G(x_1 - x, t)\right) G(x_0 - x, t)\, \mathrm{d}t\mathrm{d}x \\
&\qquad\qquad = \frac{(-1)^{(n-2)/2}}{2^{n+1}\pi^{n-1} |x_1 - x_0|^{n-1}} \left(\partial_s^{n}\mathcal{H}_s\mathcal{R}\chi_\Omega\right)(\omega_\star, s_\star)\ ,
\end{aligned}$$

which is the equality claimed in (3.4).

Now recall the definition of $\mathcal{B}_\Omega$ (see Equation (2.5)) as well as the explicit representation (3.1) for the fundamental solution of the wave equation in even dimension. Further notice that for any integer ν, the formal L^2 adjoint of $\mathcal{D}_t^\nu$ is given by $(\mathcal{D}_t^\nu)^* = (-1)^\nu t\mathcal{D}_t^\nu t^{-1}$. We therefore can compute

$$\begin{aligned}
&(\mathcal{B}_\Omega \mathcal{W} f)(x_0) \\
&= \frac{1}{\pi^{n/2}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_0^\infty \mathcal{D}_t^{(n-2)/2} \left(\frac{\chi\left\{t^2 > |x_0 - x|^2\right\}}{\sqrt{t^2 - |x_0 - x|^2}} \right) \mathcal{W} f(x, t)\ \mathrm{d}t\, \mathrm{d}S(x) \\
&= \frac{(-1)^{(n-2)/2}}{\pi^{n/2}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_0^\infty \frac{\chi\left\{t^2 > |x_0 - x|^2\right\}}{\sqrt{t^2 - |x_0 - x|^2}} \left(t\mathcal{D}_t^{(n-2)/2} t^{-1}\mathcal{W} f\right)(x, t)\ \mathrm{d}t\, \mathrm{d}S(x) \\
&= \frac{(-1)^{(n-2)/2}}{\pi^{n/2}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_{|x_0 - x|}^\infty \frac{(t\mathcal{D}_t^{(n-2)/2} t^{-1}\mathcal{W} f)(x, t)}{\sqrt{t^2 - |x_0 - x|^2}}\, \mathrm{d}t\, \mathrm{d}S(x)\ .
\end{aligned}$$

In fact, the second equality follows from repeated integration by parts. The boundary terms at ∞ vanish since, due to the compact support of f, all derivatives of $\mathcal{W} f(x, t)$ tend to zero as $t \to \infty$ (uniformly with respect to x). In view of Theorem 2.3 and Equation (3.4), the above expression for $(\mathcal{B}_\Omega \mathcal{W} f)(x_0)$ yields the first identity in Theorem 3.1, formula (3.2).

Finally, we verify the second identity in Theorem 3.1. Interchanging the order of differentiation in the last displayed expression for $(\mathcal{B}_\Omega \mathcal{W} f)(x_0)$ followed by one integration by parts yields

$$
\begin{aligned}
&(-1)^{(n-2)/2}\,\pi^{-n/2}\left(\mathcal{B}_\Omega \mathcal{W} f\right)(x_0) \\
&\quad = -\int_{\partial\Omega} \nu_x \cdot (x_0 - x) \int_0^\infty \partial_t \left(\frac{\chi\left\{t^2 > |x_0 - x|^2\right\}}{\sqrt{t^2 - |x_0 - x|^2}} \right) \mathcal{D}_t^{(n-2)/2} t^{-1} \mathcal{W} f(x,t)\, \mathrm{d}t \mathrm{d}S(x) \\
&\quad = \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \int_{|x_0 - x|}^\infty \frac{\left(\partial_t \mathcal{D}_t^{(n-2)/2} t^{-1} \mathcal{W} f\right)(x,t)}{\sqrt{t^2 - |x_0 - x|^2}}\, \mathrm{d}t\, \mathrm{d}S(x)\ .
\end{aligned}
$$

This shows equation (3.3) and concludes the proof of Theorem 3.1.

3.2. Proof of Theorem 1.2. Recall the formula (2.2) for the solution of the wave equation (2.1) as well as the explicit expression (3.1) for the fundamental solution of the wave equation in even dimensions. After introducing polar coordinates around the center $x \in \partial\Omega$ we can write

$$
\begin{aligned}
(\mathcal{W} f)(x,t) &= \int_\Omega \left(\partial_t G(x_1 - x, t)\right) f(x_1)\, \mathrm{d}x_1 \\
&= \frac{1}{2\pi^{n/2}} \int_\Omega \left(\partial_t \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 - |x_1 - x|^2 > 0\right\}}{\sqrt{t^2 - |x_1 - x|^2}} \right) f(x_1)\, \mathrm{d}x_1 \\
&= \frac{\omega_{n-1}}{2\pi^{n/2}} \int_0^\infty r^{n-1} \mathcal{M} f(x,r) \left(\partial_t \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 - r^2 > 0\right\}}{\sqrt{t^2 - r^2}} \right) \mathrm{d}r\,.
\end{aligned}
$$

After multiplying the last displayed equation with $G(x_0 - x, t)$, integrating over the time variable and using the shorthand notation $R_0 = |x_0 - x|$ we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} G(x_0 - x, t)\, \mathcal{W} f(x,t)\, \mathrm{d}t \\
&\quad = \frac{\omega_{n-1}}{4\pi^n} \int_0^\infty \left(\mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 > R_0^2\right\}}{\sqrt{t^2 - R_0^2}} \right) \times \\
&\qquad\qquad \int_0^\infty r^{n-1} \mathcal{M} f(x,r) \left(\partial_t \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 > r^2\right\}}{\sqrt{t^2 - r^2}} \right) \mathrm{d}r \mathrm{d}t \\
&\quad = \frac{\omega_{n-1}}{4\pi^n} \int_0^\infty r^{n-1} \mathcal{M} f(x,r)\ \times \\
&\qquad\qquad \int_0^\infty \left(\mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 > R_0^2\right\}}{\sqrt{t^2 - R_0^2}} \right) \left(\partial_t \mathcal{D}_t^{(n-2)/2} \frac{\chi\left\{t^2 > r^2\right\}}{\sqrt{t^2 - r^2}} \right) \mathrm{d}t \mathrm{d}r \\
&\quad = -\frac{\omega_{n-1}}{4\pi^n} \int_0^\infty r^{n-1} \mathcal{M} f(x,r)\, \mathcal{D}_{R_0}^{(n-2)/2} \mathcal{D}_r^{(n-2)/2} \times \\
&\qquad\qquad \left(\mathcal{D}_r \lim_{T \to \infty} \int_{\max\{R_0, r\}}^T \frac{2t \mathrm{d}t}{\sqrt{t^2 - R_0^2}\sqrt{t^2 - r^2}} \right) \mathrm{d}r\,.
\end{aligned}
$$

The inner integral has already been computed (see Equation (3.5)) and shows

$$
\mathcal{D}_r \lim_{T \to \infty} \int_{\max\{R_0, r\}}^T \frac{2t \mathrm{d}t}{\sqrt{t^2 - R_0^2}\sqrt{t^2 - r^2}} = -\frac{1}{r^2 - R_0^2} = -\Phi\left(r^2 - R_0^2\right)\ ,
$$

with Φ denoting the principal value distribution P.V. $1/s$. After recalling that $\mathcal{D}_r$ denotes differentiation with respect to r^2 and that the formal L^2 adjoint of $\mathcal{D}_r^\nu$ is given by $(\mathcal{D}_r^\nu)^* = (-1)^\nu \, r\mathcal{D}_r^\nu r^{-1}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} G\left(x_0 - x, t\right) \mathcal{W}f\left(x, t\right) \mathrm{d}t \\
&\qquad = \frac{(-1)^{(n-2)/2}\,\omega_{n-1}}{4\pi^n} \int_0^\infty r^{n-1} \mathcal{M}f\left(x, r\right) \mathcal{D}_r^{n-2} \Phi\left(r^2 - \left|x_0 - x\right|^2\right) \mathrm{d}r \\
&\qquad = \frac{(-1)^{(n-2)/2}\,\omega_{n-1}}{4\pi^n} \int_0^\infty \frac{\left(r\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M}f\right)(x, r)}{r^2 - \left|x_0 - x\right|^2} \mathrm{d}r \,.
\end{aligned}$$

Finally, by using the definition of $\mathcal{B}_\Omega$ (see Equation (2.5)) and inserting the identity just established we obtain

$$\begin{aligned}
\left(\mathcal{B}_\Omega \mathcal{W}f\right)(x_0) &= 2\,\nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_{\mathbb{R}} G\left(x_0 - x, t\right) \mathcal{W}f\left(x, t\right) \mathrm{d}t \mathrm{d}S\left(x\right) \\
&= \frac{(-1)^{(n-2)/2}\,\omega_{n-1}}{2\pi^n} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_0^\infty \frac{\left(r\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M}f\right)(x, r)}{r^2 - \left|x_0 - x\right|^2} \mathrm{d}r \mathrm{d}S\left(x\right) \,.
\end{aligned} \tag{3.8}$$

According to Theorem 2.3 and Equation (3.4) this shows the first inversion formula (1.5) in Theorem 1.2.

It remains to verify Equation (1.6). This equation, however, is an easy consequence of the identity (1.5) just established. In fact, interchanging the order of integration and differentiation in (3.8) and integrating by parts yields

$$\begin{aligned}
\left(\mathcal{B}_\Omega \mathcal{W}f\right)(x_0) = {} & \frac{(-1)^{(n-2)/2}\,\omega_{n-1}}{2\pi^n} \times \\
& \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \int_0^\infty \frac{\left(\partial_r \mathcal{D}_r^{n-2} r^{n-2} \mathcal{M}f\right)(x, r)}{r^2 - \left|x_0 - x\right|^2} \mathrm{d}r \mathrm{d}S\left(x\right) \,.
\end{aligned}$$

Again, according to Theorem 2.3 and Equation (3.4) the last displayed equation implies (1.6) and concludes the proof of Theorem 1.2.

4. Inversion in odd dimension. Now let $n \geq 3$ be an odd natural number. In this case, the outgoing fundamental solution of the wave equation is given

$$G\left(x, t\right) = \begin{cases} \frac{1}{2\pi^{(n-1)/2}} \, \mathcal{D}_t^{(n-3)/2} \delta\left(t^2 - |x|^2\right) & \text{on } \{t > 0\} \\ 0 & \text{on } \{t < 0\} \end{cases} . \tag{4.1}$$

Here, as usual, the operators $\mathcal{D}_t = (2t)^{-1}\,\partial_t$ denotes the distributional derivative with respect to the variable t^2.

We have the following counterpart of Theorem 3.1 for odd dimensions.

THEOREM 4.1 (Wave inversion in odd dimension). *Let $n \geq 3$ be an odd natural number, let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function that is supported inside Ω.*

Then, for every $x_0 \in \Omega$, we have

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-3)/2}}{2\pi^{(n-1)/2}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_t^{(n-3)/2} t^{-1} \mathcal{W} f\right)(x, |x_0 - x|) \,\mathrm{d}S(x) , \tag{4.2}$$

$$f(x_0) = (\mathcal{K}_\Omega f)(x_0) + \frac{(-1)^{(n-3)/2}}{2\pi^{(n-1)/2}} \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \left(\partial_t \mathcal{D}_t^{(n-3)/2} t^{-1} \mathcal{W} f\right)(x, |x_0 - x|) \,\mathrm{d}S(x) . \tag{4.3}$$

Here, again, $\mathcal{K}_\Omega$, ν_x, ∇_{x_0}, and $\mathrm{d}S$ are as in Theorem 1.2.

We proceed with this section by first establishing Theorem 4.1 and then deriving the formulas in Theorem 1.3 as corollaries of it.

4.1. Proof of Theorem 4.1. Similar to the even dimensional case we apply Theorem 2.3 and verify that the kernel $k_\Omega^{(2)}$ defined in (2.7) is equal to the kernel k_Ω defined in (1.4), and that $(\mathcal{B}_\Omega \mathcal{W} f)(x_0)$ can be written as any of the integral terms in Equations (4.2) and (4.3).

We first show that $k_\Omega^{(2)} = k_\Omega$, that is,

$$(\nabla_{x_0} + \nabla_{x_1})^2 \int_\Omega \int_{\mathbb{R}} (\partial_t G(x_1 - x, t))\, G(x_0 - x, t) \,\mathrm{d}t\mathrm{d}x = \frac{(-1)^{(n-1)/2}}{2^{n+1}\pi^{n-1} |x_1 - x_0|^{n-1}} (\partial_s^n \mathcal{R}\chi_\Omega)(\omega_\star, s_\star) , \tag{4.4}$$

where $\omega_\star = \omega_\star(x_0, x_1) = \frac{x_1 - x_0}{|x_1 - x_0|}$ and $s_\star = s_\star(x_0, x_1) = \frac{|x_1|^2 - |x_0|^2}{2|x_1 - x_0|}$ are as in (1.2).

With the notation $R_0 := |x_0 - x|$ and $R_1 := |x_1 - x|$, the representation (4.1) for the outgoing fundamental solution of the wave equation in odd dimensions yields

$$\begin{aligned}
\int_{\mathbb{R}} (\partial_t G(x_1 - x, t))\, G(x_0 - x, t) \,\mathrm{d}t
&= \frac{1}{4\pi^{n-1}} \int_0^\infty \left(\partial_t \mathcal{D}_t^{(n-3)/2} \delta\left(t^2 - R_1^2\right)\right) \mathcal{D}_t^{(n-3)/2} \delta\left(t^2 - R_0^2\right) \mathrm{d}t \\
&= \frac{2}{4\pi^{n-1}} \int_0^\infty \left(t \mathcal{D}_t^{(n-1)/2} \delta\left(t^2 - R_1^2\right)\right) \mathcal{D}_t^{(n-3)/2} \delta\left(t^2 - R_0^2\right) \mathrm{d}t \\
&= -\frac{2}{4\pi^{n-1}} \int_0^\infty \left(t \mathcal{D}_{R_1}^{(n-1)/2} \delta\left(t^2 - R_1^2\right)\right) \mathcal{D}_{R_0}^{(n-3)/2} \delta\left(t^2 - R_0^2\right) \mathrm{d}t \\
&= -\frac{1}{4\pi^{n-1}} \mathcal{D}_{R_1}^{(n-1)/2} \mathcal{D}_{R_0}^{(n-3)/2} \int_0^\infty \delta\left(t^2 - R_1^2\right) \delta\left(t^2 - R_0^2\right) 2t \,\mathrm{d}t \\
&= -\frac{1}{4\pi^{n-1}} \mathcal{D}_{R_1}^{(n-1)/2} \mathcal{D}_{R_0}^{(n-3)/2} \delta\left(R_0^2 - R_1^2\right) \\
&= \frac{(-1)^{(n-3)/2}}{4\pi^{n-1}} \delta^{(n-2)} \left(|x_0 - x|^2 - |x_1 - x|^2\right) .
\end{aligned}$$

Here and in the following $\delta^{(\nu)}$ denotes the ν-th derivative of the one-dimensional delta distribution for some integer number $\nu \geq 0$.

Now recall the definitions $\omega_\star = \frac{x_1 - x_0}{|x_1 - x_0|}$ and $s_\star = \frac{|x_1|^2 - |x_0|^2}{2|x_1 - x_0|}$ and write $x = s\omega_\star + y$ with $s \in \mathbb{R}$ and $y \perp \omega$. We then have (see Equation (3.7))

$$|x_0 - x|^2 - |x_1 - x|^2 = 2\,|x_1 - x_0|\,(s - s_\star) .$$

This implies

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}} \left(\partial_t G\left(x_1 - x, t\right)\right) G\left(x_0 - x, t\right) \mathrm{d}t \mathrm{d}x \\
&\qquad = \frac{(-1)^{(n-3)/2}}{4\pi^{n-1}} \int_{\mathbb{R}} \int_{\omega_\star^\perp} \chi_\Omega\left(s\omega_\star + y\right) \delta^{(n-2)}\left(2\left|x_1 - x_0\right|\left(s - s_\star\right)\right) \mathrm{d}y \mathrm{d}s \\
&\qquad = \frac{(-1)^{(n-3)/2}}{4\pi^{n-1}} \int_{\mathbb{R}} \delta^{(n-2)}\left(2\left|x_1 - x_0\right|\left(s - s_\star\right)\right) \left(\int_{\omega_\star^\perp} \chi_\Omega\left(s\omega_\star + y\right) \mathrm{d}y \right) \mathrm{d}s \\
&\qquad = \frac{(-1)^{(n-3)/2}}{4\pi^{n-1}} \int_{\mathbb{R}} \delta^{(n-2)}\left(2\left|x_1 - x_0\right|\left(s - s_\star\right)\right) \left(\mathcal{R}\chi_\Omega\right)\left(\omega_\star, s\right) \mathrm{d}s \,.
\end{aligned}
$$

Integrating $n-2$ times by parts yields

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}} \left(\partial_t G\left(x_1 - x, t\right)\right) G\left(x_0 - x, t\right) \mathrm{d}t \mathrm{d}x \\
&\quad = \frac{(-1)^{(n-1)/2}}{4\pi^{n-1}} \frac{1}{2^{n-2}\left|x_1 - x_0\right|^{n-2}} \int_{\mathbb{R}} \delta\left(2\left|x_1 - x_0\right|\left(s - s_\star\right)\right) \left(\partial_s^{n-2} \mathcal{R}\chi_\Omega\right)\left(\omega_\star, s\right) \mathrm{d}s \\
&\quad = \frac{(-1)^{(n-1)/2}}{2^{n+1}\pi^{n-1}\left|x_1 - x_0\right|^{n-1}} \left(\partial_s^{n-2} \mathcal{R}\chi_\Omega\right)\left(\omega_\star, s_\star\right) \,.
\end{aligned}
$$

As in the even dimension case, after application of $\left(\nabla_{x_0} + \nabla_{x_1}\right)^2$ this yields (4.4).

Next, note that the fundamental solution (4.1) in odd dimensions may be rewritten in the form $1/(4\pi^{(n-1)/2}) \mathcal{D}_t^{(n-3)/2} t^{-1} \delta\left(t - |x|\right)$. Consequently, by the definition of $\mathcal{B}_\Omega$ (see Equation (2.5)), we have

$$
\begin{aligned}
\left(\mathcal{B}_\Omega \mathcal{W} f\right)\left(x_0\right) &= 2\,\nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_{\mathbb{R}} G\left(x - x_0, t\right) \mathcal{W} f\left(x, t\right) \mathrm{d}t \mathrm{d}S\left(x\right) \\
&= \frac{(-1)^{(n-3)/2}}{2\pi^{(n-1)/2}}\, \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_t^{(n-3)/2} t^{-1} \mathcal{W} f\right)\left(x, \left|x_0 - x\right|\right) \mathrm{d}S\left(x\right) \,.
\end{aligned}
$$

In view of Theorem 2.3 and due to Equation (4.4) this implies formula (4.2) claimed in Theorem 4.1. Finally, carrying out the differentiation under the integral yields

$$
\begin{aligned}
&\left(\mathcal{B}_\Omega \mathcal{W} f\right)\left(x_0\right) \\
&\qquad = \frac{(-1)^{(n-3)/2}}{2\pi^{(n-1)/2}} \int_{\partial\Omega} \nu_x \cdot \left(x_0 - x\right) \left(\partial_t \mathcal{D}_t^{(n-3)/2} t^{-1} \mathcal{W} f\right)\left(x, \left|x_0 - x\right|\right) \mathrm{d}S\left(x\right) \,.
\end{aligned}
$$

This shows that also identity (4.3) holds and concludes the proof of Theorem 4.1.

4.2. Proof of Theorem 1.3. Inserting the expression (4.1) for the fundamental solution of the wave equation in odd dimensions (see Equation (2.2)) shows that the solution of the wave equation (2.1) can be written as

$$
\mathcal{W} f\left(x, r\right) = \frac{\omega_{n-1}}{4\pi^{(n-1)/2}}\, \partial_r \mathcal{D}_r^{(n-3)/2} r^{n-2} \mathcal{M} f\left(x, r\right) \,.
$$

Together with the definition of $\mathcal{B}_\Omega$ (see Equation (2.5)) this yields

$$\begin{aligned}
&(\mathcal{B}_\Omega \mathcal{W} f)(x_0) \\
&\quad = \frac{1}{2\pi^{(n-1)/2}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_r^{(n-3)/2} r^{-1} \mathcal{W} f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \\
&\quad = \frac{\omega_{n-1}}{8\pi^{n-1}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_r^{(n-3)/2} r^{-1} \partial_r \mathcal{D}_r^{(n-3)/2} r^{n-2} \mathcal{M} f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \\
&\quad = \frac{\omega_{n-1}}{4\pi^{n-1}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \ .
\end{aligned}$$

According to Theorem 2.3 and Equation (4.4) this yields (1.7).

It remans to establish the second formula in Theorem 1.3, namely Equation (1.8). To that end, one simply carries out the differentiation in the last displayed formula for $(\mathcal{B}_\Omega \mathcal{W} f)(x_0)$, which yields

$$\begin{aligned}
(\mathcal{B}_\Omega \mathcal{W} f)(x_0) &= \frac{\omega_{n-1}}{4\pi^{n-1}} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \left(\mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \\
&= \frac{\omega_{n-1}}{4\pi^{n-1}} \int_{\partial\Omega} \nu_x \cdot (x_0 - x) \left(\partial_r \mathcal{D}_r^{n-2} r^{n-2} \mathcal{M} f\right)(x, |x_0 - x|) \, \mathrm{d}S(x) \ .
\end{aligned}$$

This however yields formula (1.8).

5. Exact inversion for elliptical domains. Let $A = \operatorname{diag}(a_1, \dots, a_n)$ be a diagonal matrix in $\mathbb{R}^{n \times n}$ with entries $a_j > 0$ and consider the elliptical domain

$$\Omega := \left\{ x \in \mathbb{R}^n : \left|A^{-1} x\right|^2 < 1 \right\} .$$

In order to establish the exact inversion formulas of Theorem 1.4 it is sufficient to show that $\mathcal{K}_\Omega f = 0$. This will be done by first verifying that the kernel vanishes for the special case that the domain is a ball and then applying a linear transformation to the ball to establish the result for general case.

5.1. Special case: spherical domains. Let $B := \{x \in \mathbb{R}^n : |x| < 1\}$ denote the unit ball in $\mathbb{R}^n$ centered at the origin. Then, elementary geometry shows that the Radon transform of χ_B is given by

$$\mathcal{R}\chi_B(\omega, s) = \begin{cases} V_{n-1} \left(1 - s^2\right)^{(n-1)/2} & \text{if } |s| < 1 \\ 0 & \text{otherwise} \end{cases} , \tag{5.1}$$

where $V_{n-1} = \frac{\omega_{n-2}}{n-1}$ denotes the volume of the unit ball in $\mathbb{R}^{n-1}$.

Odd dimension. If n is odd, then (5.1) shows that $\mathcal{R}\chi_B(\omega, s)$ is a polynomial of degree $n-1$ on $\{|s| < 1\}$. Since $|s_\star(x_0, x_1)| < 1$ for any two distinct points $x_0, x_1 \in B$, this yields

$$k_B(x_0, x_1) = \frac{(-1)^{(n-1)/2}}{2^{n+1}\pi^{n-1}} \frac{(\partial_s^n \mathcal{R}\chi_B)(\omega_\star(x_0, x_1), s_\star(x_0, x_1))}{|x_1 - x_0|^{n-1}} = 0 \quad \text{for } x_1 \neq x_0 \in B .$$

This implies that we also have $\mathcal{K}_B f = 0$ and, according to Theorem 1.3, this shows the inversion formulas (1.11), (1.12) stated in Theorem 1.4 for the special case that the considered domain is a ball in odd dimension.

Even dimension. If $n \geq 2$ is an even natural number, then the identity $\mathcal{K}_B f = 0$ is slightly less obvious. In this case, we first note the following identity (see, for example, [21, Table 7.3, Number 13])

$$(\mathcal{H}_s (s\phi))(\hat{s}) = \hat{s} (\mathcal{H}_s \phi)(\hat{s}) - \int_{\mathbb{R}} \phi(s) \, \mathrm{d}s \tag{5.2}$$

satisfied by the Hilbert transform and some function $\phi\colon \mathbb{R} \to \mathbb{R}$. Further, Equation (5.1) shows that we have the relation $(\mathcal{R}\chi_B)(\omega, s) = P_{n-2}(s) \phi_{1/2}(s)$, where $P_{n-2}(s)$ is a polynomial of degree $n-2$ and $\phi_{1/2}(s) := \sqrt{\max\{0, 1 - s^2\}}$. Applying the identity (5.2) repeatedly, thus yields

$$(\mathcal{H}_s \mathcal{R}\chi_B)(\omega, s) = Q_{n-2}(s) \left(\mathcal{H}_s \phi_{1/2}\right)(s) + Q_{n-3}(s) \,,$$

for certain polynomials Q_{n-2} and Q_{n-3} of degree $n-2$ and $n-3$, respectively. The Hilbert transform of $\phi_{1/2}(s) = \sqrt{\max\{0, 1 - s^2\}}$ is known and given by (see, for example, [4, Table 13.11])

$$\left(\mathcal{H}_s \phi_{1/2}\right)(s) = -s + \operatorname{sign}(s) \, \chi\{|s| > 1\} \sqrt{s^2 - 1} \qquad \text{for all } s \in \mathbb{R} \,.$$

In particular, $\left(\mathcal{H}_s \phi_{1/2}\right)(s)$ is a linear function on $\{|s| < 1\}$ and therefore the product $Q_{n-2}(s) (\mathcal{H}_s \phi_{1/2})(s)$ is a polynomial of degree $n-1$ on $\{|s| < 1\}$. Noting again that $|s_\star(x_0, x_1)| < 1$ for any two distinct points $x_0, x_1 \in B$, we therefore conclude

$$k_B(x_0, x_1) = \frac{(-1)^{(n-2)/2}}{2^{n+1} \pi^{n-1}} \frac{(\partial_s^n \mathcal{H}_s \mathcal{R}\chi_B)(\omega_\star(x_0, x_1), s_\star(x_0, x_1))}{|x_1 - x_0|^{n-1}} = 0 \quad \text{for } x_1 \neq x_0 \in B \,.$$

This implies that the identity $\mathcal{K}_B f = 0$ also holds in even dimension. In view of Theorem 1.2, this establishes inversion formulas (1.9), (1.10) in Theorem 1.4 for the special case that the considered domain is a ball in even dimension.

5.2. General case: elliptical domains. Now let $\Omega = \left\{x \in \mathbb{R}^n : |A^{-1}x| < 1\right\}$ be an elliptical domain where $A = \operatorname{diag}(a_1, \dots, a_n)$ is a diagonal matrix with positive entries that possibly differ from each other. We then obviously have the identity $\chi_\Omega(x) = \chi_B\left(A^{-1}x\right)$, where $B \subset \mathbb{R}^n$ is the unit ball considered in the previous subsection. Therefore, the known relation between the Radon transform of a function φ and the Radon transform of the function $x \mapsto \varphi\left(A^{-1}x\right)$ implies that

$$\mathcal{R}\chi_\Omega(\omega, s) = \frac{\det(A)}{|A\omega|} \mathcal{R}\chi_B\left(\frac{A\omega}{|A\omega|}, \frac{s}{|A\omega|}\right) \qquad \text{for all } (\omega, s) \in S^{n-1} \times \mathbb{R} \,, \tag{5.3}$$

From (5.3) we conclude that

$$\begin{aligned}
(\partial_s^n \mathcal{R}\chi_\Omega)(\omega, s) &= \frac{\det(A)}{|A\omega|^{n+1}} \, (\partial_s^n \mathcal{R}\chi_B)\left(\frac{A\omega}{|A\omega|}, \frac{s}{|A\omega|}\right) && \text{if } n \text{ is odd}\,, \\
(\partial_s^n \mathcal{H}_s \mathcal{R}\chi_\Omega)(\omega, s) &= \frac{\det(A)}{|A\omega|^{n+1}} \, (\partial_s^n \mathcal{H}_s \mathcal{R}\chi_B)\left(\frac{A\omega}{|A\omega|}, \frac{s}{|A\omega|}\right) && \text{if } n \text{ is even}\,.
\end{aligned}$$

According to the special case considered in Subsection 5.1 this shows that $k_\Omega(x_0, x_1) = 0$ for all $x_0 \neq x_1 \in \Omega$ and hence that $\mathcal{K}_\Omega f = 0$. In view of Theorems 1.2 and 1.3, this establishes the exact reconstruction formulas (1.9), (1.10), (1.11) and (1.12) for the inversion of spherical means on elliptical domains in arbitrary spatial dimension.

6. Discussion. Many medical imaging and remote sensing applications aim for recovering a function from spherical means centered on a set of admissible receiver or detector locations. In the case that the center set is an infinite hyperplane explicit formulas of the back-projection type for recovering a function from spherical means are known since the mid 80s (see [1, 6]). In the case that center set is a spherical or cylindrical surface such type of formulas have been derived about 20 years later in [7, 8, 13, 24]. All these geometries have rotational and/or translational invariance and seem well adapted to the inversion from spherical means. It therefore has been believed by many researchers that such exact back-projection type inversion formulas may only exist for those invariant geometries.

Very recently, in [11, 15] explicit exact inversion formulas of the back-projection type for inverting the spherical mean transform with elliptical center sets in dimensions $n = 2$ and $n = 3$ have been derived (see [19] for a different formula for ellipsoids in arbitrary dimension and [14] for reconstruction formulas for certain polygons and polyhedra). Moreover, in [11, 15] it has been shown that the same formulas may be applied when the center set equals the boundary of an arbitrarily shaped smooth convex domain Ω, in which case these formulas recover the unknown function modulo an explicitly computed integral operator $\mathcal{K}_\Omega$. In the present paper we generalize these results to the case of arbitrary spatial dimension. We have further shown, that the operator $\mathcal{K}_\Omega$ vanishes for elliptical domains which yields exact inversion formulas in these cases. However, as can be readily verified by using Equation (1.4), the operator $\mathcal{K}_\Omega$ does not vanish for general domains. Actually, these results give an affirmative negative answer to the question whether the universal back-projection formula of Xu and Wang [24] (introduced there for the case $n = 3$) is exact for general domains. This negative result, however, does not imply that a different back-projection type formula may provide exact reconstruction for general domains. Nevertheless, it is believed by the author that such a *truly universal reconstruction formula* for the spherical mean transform does not exist.

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